QUALITATIVE PROPERTIES OF SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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QUALITATIVE PROPERTIES OF SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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PYDA SRINIVAS

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Dedicated

to

My Parents

Smt. P.V. Rajeswari

and

Sri P. Babu Rao

18/7/83
BP

CERTIFICATE

This is to certify that the work embodied in the thesis entitled 'QUALITATIVE PROPERTIES OF SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS' being submitted by Pyda Srinivas has been carried out under my supervision. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

July, 1983.

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CONTENTS

SYNOPSIS

A thesis entitled "Qualitative Properties of Solutions of Volterra Integro-Differential Equations" is submitted in partial fulfilment of the requirement for the Ph.D. degree by Pyda Srinivas to the Department of Mathematics, Indian Institute of Technology, Kanpur.

Quite often linear and nonlinear Volterra integro-differential equations arise in diverse areas such as physical, biological, social and engineering sciences. The way such equations arise and the properties of solutions of these equations to various physical phenomena have been investigated by many scientists. However, until quite recently, the attempts have not been made to investigate the qualitative properties such as boundedness, oscillatory and stability of these equations. The object of this thesis is to study the positivity, boundedness, eventual stability and L^p-stability properties of solutions of Volterra integro-differential equations. Dichotomy properties of linear equations also find a place in this thesis.

Friedman, Komlenko, Levin, Miller and Padmavalli among others have studied the positivity and boundedness of solutions of Volterra integral equations and many interesting results have been reported. In the case of integral equations of the form

$$x(t) = f(t) - \int_0^t a(t,s)\, x(s)ds, \quad 0 \leq t < \infty \qquad (1)$$

one can assert by virtue of Neumann series that all the solutions of (1) are positive in their interval of existence, if $a(t,s)$ is negative and $f(t)$ is positive. However, the same is not true in general for integro-differential equations of the form

$$x'(t) = f(t) - \int_0^t a(t,s)x(s)\, ds, \quad x(0) = x_0 > 0 \qquad (2)$$

particularly when $a(t,s)$ is positive. Sufficient conditions for positivity and boundedness of solutions of (2) are investigated in the case where $a(t,s)$ is positive. Our approach here involves the construction of a scalar function which in certain cases turns out to be the usual resolvent kernel. Examples and remarks are provided at appropriate places to indicate the advantage of this approach and illustrate the generality of conditions.

Eventual uniform asymptotic stability is a natural generalisation of uniform asymptotic stability of a given differential equation in which it is not assumed that the zero function is a solution of it. This situation would in general happen for integro-differential equations of the form

$$x'(t) = f(t) + \int_0^t a(t,s)G(s,x(s))ds \qquad (3)$$

When either $f(t) \not\equiv 0$ or $G(t,0) \not\equiv 0$. In the later part of the thesis, we shall investigate sufficient conditions for

eventual uniform asymptotic stability of the set $\{0\}$ for (3) when $G(s,x)$ is linear in x. The results thus obtained are used to study the eventual properties of solutions of the nonlinear equation (3) under more general class of non-linear functions G.

Liapunov second method involving the construction of a scalar function is one of the powerful tools in the qualitative study of ordinary differential equations and Volterra integro-differential equations. Concerning the application of this method for functional differential equations, essentially two approaches have emerged in recent years such as the construction of a Liapunov (i) function and (ii) functional. The Liapunov-Razumikhin technique is basically of the former type and this has been adapted to discuss the L^2-stability properties of solutions of Volterra integro-differential equations.

The study of linear analysis plays a vital role to understand the complexity of the corresponding nonlinear analysis. Most of the authors in their recent studies on linear Volterra integro-differential equations either by Liapunov second method or from perturbation analysis, have assumed that the coefficient matrix is stable. Notable exceptions that have dispensed with this condition on the coefficient matrix have been the investigations of Levin, Grossman and Miller, Grimmer and Seifert, and Burton among others. Motivated

from the interesting nature of the problem, an attempt has been made in this thesis to study exponential stability properties of solutions of Volterra integro-differential systems when the coefficient matrix is not necessarily stable. Our approach here has been to develop an equivalent equation which involves an arbitrary function. The proper choice of this function would pave a way for the new coefficient matrix (corresponding to equivalent Volterra integro-differential equation) to be stable. The stability analysis of linear autonomous systems basically characterises the sign of the real parts of the characteristic roots of the coefficient matrix. However, this approaches does not carry over to non-autonomous systems. Thus the study of dichotomies of solutions of linear non-autonomous systems assumes a prominent role. In the last chapter of this thesis the dichotomy properties of linear Volterra integro-differential systems are discussed.

CHAPTER - 1

INTRODUCTION

1.1 Historical Notes

Ever since the beginning of calculus, many physical phenomena have been modelled as equations involving derivatives or integrals. Naturally as a follow-up of any modelling, it is necessary that one should know about the solutions of these equations. During the initial decades of its existence, this theory consisted only of isolated methods of solving certain type of linear equations. The problem of existence of a solution and its explicit representation was posed already in the eighteenth century itself. Numerous investigations revealed that the explicit representation of the solutions is an extremely rare phenomenon and in many cases (particularly for linear equations with variable coefficients and nonlinear equations), it is rather difficult to obtain the same in the explicit form. Further, the methods of numerical integration of these equations did not open the horizons for the development of the general theory since these methods yield only one particular solution that too on a finite interval. Applications - in particular to the problems of celestial mechanics, bio-mechanics, bio-mathematics and in social and engineering sciences - require the clarification of at least the nature of the behaviour of

solutions of these equations in the entire domain of their existence without explicitly solving them. In this direction. the development of the qualitative theory owes its legacy to Poincare and Liapunov towards the end of last century, forms a significant contribution. The basic problems of study of this theory are (i) the existential analysis and (ii) the properties of solutions such as positivity, boundedness and stability.

The qualitative theory of integral equations largely owes its origin to Vito Volterra and Ivar Fredholm. In 1896 Volterra encountered integro-differential equations while investigating the dynamical processes mainly in biology. With the advent of Fredholm theory in 1900 and with its tremendous impact on mathematical and functional, analysis, the study of Volterra integral equations was relegated into the background for quite some time. However, during the last thirty years Volterra equations have emerged vigourously in many applied fields such as nuclear reactor dynamics (cf. [28]), automatic control systems (cf. [8],[9],[12],[26]), the theory of industrial inventory (cf. [38]), and in biology and hereditary mechanics.

Classically there are three methods available for the qualitative study of these equations viz., Perron's technique, Liapunov direct method and Popov's method. Perron's technique basically involves the study of perturbed

linear systems when the perturbation functions are sufficiently small in a certain sense. This technique has been employed extensively by various authors (cf. [16],[17],[33],[35],[36],[37]) for Volterra integro-differential equations via the resolvent function and the variation of constants formula. In Popov's method, the integral transforms are used and as such it is severely restricted to integral equations of convolution type. A common feature of the systems to which Popov's method could be applied is that the kernel is integrable. This method has been exploited by various authors (cf. [8],[9],[12],[13],[31]) to study the stability properties of solutions of Volterra integral equations. Liapunov direct method involves the construction of a scalar function, the time derivative of which should satisfy a negative estimate along the solutions of the equation. For integro-differential equations, Liapunov functionals are constructed instead of functions and perhaps this is natural to expact due to the hereditary nature of Volterra equations. A number of results (cf. [1],[2],[25]) have been reported and a general result due to Levin involving non-convolution kernels could be found in [13, chapter 5.1]. In recent years Liapunov-Razumikhin technique (cf. [41],[21]) which involves functions rather than functionals has been used to study integro-differential equations by many authors (cf. [15],[24],[47],[48]) and a good number of results have been accumulated. In addition to the above techniques, the theory

of admissibility developed by Massera and Schaffer for ordinary linear differential equations has been further extended in recent years for Volterra integral equations (cf. [10],[11],[13],[34],[40]).

1.2 Brief Review

The qualitative properties of solutions of Volterra integro-differential equations have been the focus of study in many works [cf. 1-3,12,15-18,24-26,28,29,31,35,37,42,43, 47-50] under different hypotheses and various results have been accumulated.

Friedman [14], Komlenko [22], Levin [27], Ling [30], Miller [33] and Padmavalli [38] among others have studied the positivity of solutions of Volterra integral equations. While non-convolution equations are considered in [22], the convolution equations are discussed by several others [14,27, 30,33,38]. For integral equations of the form

$$x(t) = f(t) - \int_0^t a(t,s)x(s)ds, \quad 0 \leq t < \infty \tag{1.2.1}$$

an application of Neumann series yields the positivity of solutions of (1.2.1) if the kernel $a(t,s)$ is negative and the source function $f(t)$ is positive. Ling [30] has discussed the positivity of solutions of (1.2.1) in the convolution case when the kernel $a(t)$ is positive and its derivative $a'(t)$ is negative. However these results are not in general applicable to Volterra integro-differential equations.

Therefore it is interesting to investigate positivity of solutions of

$$x'(t) = f(t) - \int_0^t a(t,s)x(s)ds, \; x(0) = x_0, \; 0 \leq t < \infty \quad (1.2.2)$$

particularly when the kernel $a(t,s)$ is positive. An attempt has been made in this direction for linear and non-linear Volterra integro-differential equations.

When zero is not a solution of the system of ordinary differential equations, eventual asymptotic stability of the set $\{0\}$ is discussed in [23,51]. Very often the source function $f(t)$ is not zero in (1.2.2) and it is therefore appropriate to study the eventual properties of the set $\{0\}$ for (1.2.2). As such the eventual uniform asymptotic stability of the set $\{0\}$ for linear and non-linear Volterra integro-differential equations are discussed here using resolvent functions and variation of constants formula. Quite recently Seifert [47,48], Grimmer and Seifert [15], and Lakshmikantham and Rama Mohana Rao [24] have studied stability, boundedness, uniform stability and uniform asymptotic stability of Volterra integro-differential equations by employing Liapunov-Razumikhin technique which involves Liapunov functions rather than functionals. This technique is exploited further in this thesis to discuss L^2-stability of Volterra integro-differential systems.

The study of linear Volterra integro-differential systems (VIDS) very often requires that the coefficient matrix

be stable. However the notable exceptions that have dispensed with this condition on the coefficient matrix have been the works of Levin [25], Grossman and Miller [17], Grimmer and Seifert [15] and Burton [3]. By adopting an entirely different approach involving an arbitrary function the exponential stability properties of solutions of Volterra integro-differential equations, when the coefficient matrix is not necessarily stable, are investigated in the later part of the thesis.

1.3 Layout of the Thesis

The present thesis attempts to study some of the qualitative properties of solutions of Volterra integro-differential equations such as positivity, boundedness, eventual uniform stability, eventual uniform asymptotic stability and L^2-stability. Dichotomy properties of these equations are also discussed. The entire work of this thesis is divided into five chapters.

The first chapter deals with historical survey and a brief review of the work. Various techniques available in the literature and recent contributions towards the gradual development of the theory are indicated.

Chapter 2 deals with the positivity and boundedness of solutions of Volterra integro-differential equations. It is clear that one can assert from Neumann series that the solutions of (1.2.2) are positive if the kernel 'a' is

negative and the weight function 'f' is positive. However the same cannot be concluded if 'a' is positive. An attempt has been in this chapter to ascertain the positivity of the solutions of (1.2.2) particularly when 'a' is positive. The technique employed in this chapter is to transform (1.2.2) into an equivalent equation whose kernel would be negative by a proper choice of the auxiliary function. The results obtained here are compared with those of earlier authors and examples are provided to illustrate the effectiveness of the approach. This study includes both convolution and non-convolution equations. The case where the solutions exist over a finite time interval is also discussed.

Motivated by the fact that zero is not a solution of (1.2.2) when the forcing function is different from zero, eventual uniform, stability and asymptotic stability properties of solutions of (1.2.2) are investigated in Chapter 3. The main tools in this chapter are the resolvent kernels and variation of constants formula. Perturbed equations are also studied under a more general class of non-linear perturbations.

In Chapter 4, Liapunov-Razumikhin technique is employed to study L^2-stability properties of solutions of Volterra integro-differential systems. In order to proceed with the avowed intention, a more general result on L^p-stability of solutions of a functional differential system of delay type where the delay becomes unbounded as $t \to \infty$, is derived and then applied to Volterra integro-differential systems.

Finally in Chapter 5 the exponential asymptotic stability properties of solutions of linear Volterra integro-differential systems where the coefficient matrix is not necessarily stable, are obtained. Dichotomy properties of these systems also find a place in this chapter.

CHAPTER - 2

POSITIVITY AND BOUNDEDNESS OF SOLUTIONS

2.1 Introduction

Since long there has been a persistent effort with quite an amount of success to descern various properties - like boundedness, stability, asymptotic behaviour, positiveness, square integrability and so on - of the solutions of integral equations of Volterra type that occur in a variety of problems which are heriditary in nature such as in nuclear reactor dynamics, biological models etc. In this chapter, positivity and boundedness of solutions of Volterra Integro-differential equations are studied.

Friedman [14] , Komlenko [22], Levin [27], Rina Ling [30], Miller [33], Padmavalli [38] and Rama Mohana Rao and Srinivas [45] among others have studied the positivity and boundedness of solutions of Volterra integral equations and many interesting results have been reported. While nonconvolution kernels are taken into account by one of earlier workers [22], the kernels considered by several others [14,27,30,38] are of convolution type. The results obtained in [45] constitute the main core of this chapter. In the case of integral equations of the form

$$x(t) = f(t) - \int_0^t a(t,s)x(s)ds, \ 0 \leq t < \infty \qquad (2.1.1)$$

one can assert by virtue of Neumann series (cf. [52]) that if the kernel $a(t,s)$ is negative and the source function $f(t)$ is positive, then the solutions of (2.1.1) are positive in the interval of their existence. It is interesting to probe the conditions on $f(t)$ in order to obtain the positivity of solutions of integro-differential equation

$$x'(t) = f(t) - \int_0^t a(t,s)x(s)ds, \; x(0) = x_0 > 0 \qquad (2.1.2)$$

where $0 \leq t < \infty$, especially when the kernel $a(t,s)$ is positive. The objective of this chapter is to investigate sufficient conditions for positivity and boundedness of solutions of Volterra linear and nonlinear integro-differential equations of both convolution and nonconvolution type. Our approach herein involves the construction of an auxiliary function which, in certain cases, turns out to be the usual resolvent kernel. The advantage of such an approach is to transfer the (i) responsibility of the original kernel to a better kernel and (ii) unwanted behaviour (if any) of the original kernel to the source function, through a proper choice of an auxiliary function. Examples and remarks are provided at appropriate places to illustrate the generality of the conditions and the fruitfulness of the results.

Under suitable conditions on $a(t,s)$ (for example, $a(t,s)$ is absolutely continuous on every compact subset of $\mathbb{R} \times \mathbb{R}$), the equation (2.1.2) can be written as

$$x'(t) = f(t)-x_0 \int_0^t a(t,s)ds-\int_0^t [\int_s^t a(t,u)du]\ x'(s)ds. \quad (2.1.3)$$

Therefore, if $f(t) - x_0 \int_0^t a(t,s)ds$ is positive and $\int_s^t a(t,s)ds$ is negative, then in view of Neumann series, one can conclude that $x'(t)$ in (2.1.3) is positive and hence the solution $x(t)$ of (2.1.2) is positive as $x_0 > 0$ (such a conclusion is quite obvious if $a(t,s)$ is negative and $f(t)$ is positive). This fact has been extensively used in our subsequent analysis.

While Section 2.2 is devoted to basic results, sections 2.3 and 2.4 cover the linear non-convolution and convolution equations respectively. Non-linear equations are discussed in Section 2.5. Finally, in Section 2.6 similar results over a finite interval are obtained which assert that an heriditary system won't go 'critical' in that interval.

2.2 Basic Results

In this section we shall derive some basic results which will provide the necessary leverage to workout the planned trade off between the kernel $a(t,s)$ and the source function $f(t)$.

Lemma 2.2.1 If $a(t,s)$ is continuous on $0 \leq s \leq t < \infty$ and $f(t)$ is continuous on $0 \leq t < \infty$, then the equation (2.1.2) is equivalent to

$$y'(t) = h(t) - \int_0^t b(t,s)y(s)\,ds, \quad y(0) = x_0 > 0, \quad 0 \leq t < \infty \qquad (2.2.1)$$

where

$$\begin{aligned} b(t,s) &= a(t,s) + \bar{\varphi}_s(t,s) - \int_s^t \bar{\varphi}(t,u)a(u,s)\,du \\ &\quad + \varphi(t,t)\int_s^t a(u,s)\,du, \end{aligned} \qquad (2.2.2)$$

$\varphi(t,s)$ being a c^1-function on $0 \leq s \leq t < \infty$

and

$$\begin{aligned} h(t) &= f(t) + x_0\,\bar{\varphi}(t,t) - x_0\bar{\varphi}(t,0) + \varphi(t,t)\int_0^t f(s)\,ds \\ &\quad - \int_0^t \bar{\varphi}(t,s)\, f(s)\,ds. \end{aligned} \qquad (2.2.3)$$

Proof. Let $x(t)$ be a solution of (2.1.2) existing on the interval $0 \leq t < \infty$. Consider the identity

$$\begin{aligned} \int_0^t \varphi_s(t,s)x(s)\,ds &= \bar{\varphi}(t,t)x(t) - \bar{\varphi}(t,0)\,x_0 \\ &\quad - \int_0^t \bar{\varphi}(t,s)\,x'(s)\,ds. \end{aligned}$$

Substituting $x(t)$ and $x'(t)$ from (2.1.2) and using Fubinis theorem, we get

$$\begin{aligned} \int_0^t \varphi_s(t,s)x(s)\,ds &= x_0\,\bar{\varphi}(t,t) - x_0\,\bar{\varphi}(t,0) \\ &\quad + \bar{\varphi}(t,t)\int_0^t f(s)\,ds - \int_0^t \bar{\varphi}(t,s)f(s)\,ds \\ &\quad - \bar{\varphi}(t,t)\int_0^t \left[\int_\tau^t a(s,\tau)\,ds\right] x(\tau)\,d\tau \\ &\quad + \int_0^t \left[\int_\tau^t a(s,\tau)\bar{\varphi}(t,s)\,ds\right] x(\tau)\,d\tau \end{aligned} \qquad (2.2.4)$$

Therefore, it follows from (2.1.2), (2.2.2), (2.2.3) and (2.2.4) that

$$\int_0^t b(t,s)\, x(s)ds = -x'(t) + h(t),\ x(0) = x_0$$

This shows that every solution of (2.1.2) is a solution of (2.2.1). For converse, let y(t) be a solution of (2.2.1). Then by taking $\varphi(t,s) \equiv 1$, we have from (2.2.2) and (2.2.3), $b(t,s) = a(t,s)$ and $h(t) = f(t)$ and hence y(t) satisfies (2.1.2).

Remark 2.2.1. If we choose $b(t,s) \equiv 0$ i.e. $\varphi(t,s)$ is a solution of (2.2.2) with $b(t,s) \equiv 0$, then from (2.2.1), the solution x(t) of (2.1.2) can be obtained on integrating (2.2.3) between 0 and t.

A result similar to Lemma 2.2.1 for convolution kernels is given below.

Lemma 2.2.2. If a(t) and f(t) are continuous on $0 \leq t < \infty$, then the following equations are equivalent :

$$x'(t) = f(t) - \int_0^t a(t-s)x(s)ds,\ x(0) = x_0 \tag{2.2.5}$$

$$y'(t) = h(t) - \int_0^t b(t-s)y(s)ds,\ y(0) = x_0 \tag{2.2.6}$$

where $0 \leq t < \infty$, and

$$b(t) = a(t)+\varphi'(t)+\int_0^t \varphi(t-\tau)a(\tau)d\tau-\varphi(0)\int_0^t a(\tau)d\tau, \tag{2.2.7}$$

$\varphi \in C^1[0,\infty)$,

$$h(t) = x_0\,\bar{\varphi}(t) - x_0\,\bar{\varphi}(0) + \int_0^t \bar{\varphi}(t-\tau)f(\tau)\,d\tau$$

$$- \bar{\varphi}(0)\int_0^t f(\tau)\,d\tau + f(t) \qquad (2.2.8)$$

The following result is somewhat close to the usual resolvent equations (cf. [16] and [37]) and it is useful in the study of nonlinear Volterra integro-differential equations (see section 5).

Lemma 2.2.3. If $a(t)$ and $f(t)$ are continuous on $0 \leq t < \infty$, then every solution $x(t)$ of (2.2.5) satisfies

$$x(t) = \Psi(t) - \int_0^t c(t-\tau)x(\tau)\,d\tau \qquad (2.2.9)$$

on $0 \leq t < \infty$, where

$$c(t) = \bar{\varphi}'(t) + \int_0^t \bar{\varphi}(t-\tau)a(\tau)\,d\tau\,, \qquad (2.2.10)$$

$$\Psi(t) = x_0\,\bar{\varphi}(t) + \int_0^t f(s)\,\bar{\varphi}(t-s)\,ds, \qquad (2.2.11)$$

and $\bar{\varphi} \in C^1[0,\infty), \bar{\varphi}(0) = 1$.

Remark 2.2.2. Choose $\bar{\varphi}(t)$ (for example, $\bar{\varphi}(t) = e^{-t} + te^{-t}$ with $a(t) = e^{-2t}$) such that $c(t) \equiv 0$. Then (2.2.10) takes the form

$$\bar{\varphi}'(t) = -\int_0^t \bar{\varphi}(t-\tau)a(\tau)\,d\tau\,,\ \bar{\varphi}(0) = 1$$

Thus $\bar{\varphi}(t)$ would agree with the differential resolvent to the kernel $a(t)$ and the equation (2.2.9) gives the solution $x(t)$ of (2.1.2) on $0 \leq t < \infty$ in terms of the resolvent $\bar{\varphi}(t)$ (see Grossman and Miller [16], equations (A) and (4) with $A(t) \equiv 0$).

Remark 2.2.3.

It should be noted that the above lemmas are true for solutions defined over a finite interval $J = [0,\alpha]$ with appropriate changes.

2.3 Non Convolution Equations

In this section the positivity and boundedness properties of solutions of (2.1.2) are studied through a proper choice of the auxiliary function $\varphi(t,s)$.

Theorem 2.3.1. Suppose for $0 \le s \le t < \infty$ (i) $a(t,s) > 0$ (ii) $\lambda(t) = f(t) - x_0 \int_0^t a(t,u)du$ is an increasing function of t and (iii) $\int_0^t \int_s^t a(t,\tau)\, d\tau\, ds < 1$ hold. If $x(t)$ is any solution of (2.1.2) on $[0,\infty)$, then the following estimates are satisfied on $[0,\infty)$:

(a) $x_0 \le x(t) \le x_0 + F(t)$

(b) $0 < x'(t) \le f(t)$

where $F(t) = \int_{0-}^t f(s)ds$.

Proof. Take $\varphi(t,s) = \int_s^t a(t,u)du$. Then $\varphi(t,t) = 0$.

Since $a(t,s) > 0$, it follows from (2.2.2) that

$$b(t,s) = a(t,s)-a(t,s)-\int_s^t a(u,s)(\int_u^t a(t,\tau)d\tau)du$$
$$< 0.$$

Further, using (2.2.2) and (2.2.3) we obtain

$$h(t)-x_0\int_0^t b(t,u)\,du = f(t)+x_0\int_0^t \Big[\int_s^t a(u,s)\Big(\int_u^t a(t,\tau)\,d\tau\Big)du\Big]\,ds$$

$$-\int_0^t \Big(\int_s^t a(t,u)\,du\Big) f(s)\,ds - x_0 \int_0^t a(t,u)\,du$$

$$= f(t) - x_0 \int_0^t a(t,u)\,du$$

$$+ x_0\int_0^t \Big(\int_0^u a(u,s)\,ds\Big)\Big(\int_u^t a(t,\tau)\,d\tau\Big)du$$

$$- \int_0^t \Big(\int_s^t a(t,u)\,du\Big) f(s)\,ds$$

$$= f(t) - x_0 \int_0^t a(t,u)\,du$$

$$-\int_0^t \Big(\int_s^t a(t,\tau)\,d\tau\Big)\Big[f(s)-x_0\int_0^s a(s,u)\,du\Big]\,ds$$

$$= \lambda(t) - \int_0^t \Big[\int_s^t a(t,\tau)\,d\tau\Big]\,\lambda(s)\,ds$$

Thus the assumptions (ii) and (iii) yield $h(t)-x_0\int_0^t b(t,s)\,ds > 0$. Hence, in view of the observations made toward the end of section 1, it is clear that the solution $y(t)$ of (2.2.1) together with its derivative, is positive for all $t \in [0,\infty)$. Moreover, we have $0 < y'(t) \leq h(t)$ on $0 \leq t < \infty$. Therefore, by invoking Lemma 2.2.1, we conclude that the inequalities (a) and (b) hold for all $t \in [0,\infty)$ and this completes the proof.

Corollary 2.3.1. If, in addition to the assumptions (i), (ii) and (iii) of Theorem 2.3.1, suppose $f \in L^1[0,\infty)$, then the solution $x(t)$ of (2.1.2) is positive and bounded on $[0,\infty)$.

Example 2.3.1. Choose $a(t,s) = e^{-t}/(1+s)^3$ for $0 \leq s \leq t < \infty$. It can be seen that the conditions (i) and (iii) of Theorem 2.3.1 are satisfied. If we take $f(t) = 3-e^{-t}$ and $x_o = 1$, then the condition (ii) also holds.

Remark 2.3.1. Let

$$F_1(t) = f(t) - x_o \int_o^t a(t,u)\,du$$

and

$$K(t,s) = -\int_s^t a(t,u)\,du$$

for $0 \leq s \leq t < \infty$. Then the equation (2.1.2) reduces, upon integrating by parts (see equation (2.1.3), to

$$x'(t) = F_1(t) + \int_o^t K(t,s)\; x'(s)\; ds \qquad (2.3.1)$$

on $0 \leq s \leq t < \infty$. In order to obtain an estimate similar to (b) of Theorem 2.3.1 on the solution $x'(t)$ of (2.3.1), Komlenko [22 , Corollary 2 and remark 4] assumed the condition

$$\frac{d}{dt}\left(\frac{K(t,s)}{F_1(t)}\right) \geq 0. \qquad (2.3.2)$$

If we choose $a(t,s) = e^{-t}/(1+s)^3$ and $f(t) = 3-e^{-t}$ for $0 \leq s \leq t < \infty$, and $x_o = 1$, then

$$\left[\frac{d}{dt}\left(\frac{K(t,s)}{F_1(t)}\right)\right]_{\substack{t=0.001\\ s=0.0001}} \simeq -\,0.4975$$

Thus the condition 3.2 of Komlenko [22] does not hold. However, all the assumptions of Theorem 2.3.1 are satisfied (see Example 2.3.1).

2.4 Convolution Equations

We shall now give a result similar to Theorem 2.3.1 for convolution equation (2.2.5).

Theorem 2.4.1. Suppose for $0 \leq t < \infty$ (i) $a(t) > 0$ (ii) $a'(t) < 0$ (iii) $f(t) > 0$, $f'(0) > x_0\, a(0)$, $f''(t) > a(0)f(t)$ hold. Then if $x(t)$ is a solution of (2.2.5) on $0 \leq t < \infty$, then we have,

(a) $x_0 \leq x(t) \leq x_0 + F(t)$

(b) $0 < x'(t) \leq f(t)$

where $F(t) = \int_{0-}^{t} f(s)ds$.

Proof. Take $\varphi(t) = -\int_0^t a(\tau)d\tau$. Then from assumptions (i) and (ii) it is clear that $\varphi(0) = 0$, $\varphi'(t) = -a(t) < 0$ and $\varphi''(t) = -a'(t) > 0$. Moreover, from (2.2.7) and (2.2.8) we have

$$b(t) = -\int_0^t \left[\int_0^{t-\tau} a(s)ds\right] a(\tau)d\tau < 0 \text{ for } 0 \leq t < \infty,$$

$$h(0) = f(0) > 0 \text{ and}$$

$$h'(t) = x_0\varphi'(t)+\int_0^t \varphi'(t-\tau)f(\tau)d\tau + f'(t) \qquad (2.4.1)$$

Using the assumption (iii) and (2.4.1), we obtain

$$h'(0) = x_0\, \varphi'(0) + f'(0) = f'(0) - x_0\, a(0) > 0$$

and

$$h''(t) = x_0\, \varphi''(t)+\varphi'(0)f(t)+\int_0^t \varphi''(t-\tau)f(\tau)d\tau + f''(t)$$

$$> f''(t) - a(0)\, f(t) > 0.$$

Thus we have $h(t) > 0$ and $b(t) < 0$ for all $t \in [0,\infty)$. Hence the Lemma 2.2.2 together with the observation at the end of section 2.1 yield that the solution $x(t)$ of (2.2.5) along with its derivative $x'(t)$ are positive on $[0,\infty)$. Since $x(t) > 0$ and $a(t) > 0$, it is clear from (2.2.5) that $0 < x'(t) \leq f(t)$ and hence (a) and (b) are satisfied for all $t \in [0,\infty)$.

Remark 2.4.1. If, in addition to the assumptions of Theorem 2.4.1, suppose $f(t) \in L^1 [0,\infty)$, then the solution $x(t)$ of (2.2.5) is also bounded on $[0,\infty)$.

Remark 2.4.2. While discussing similar properties for solutions of Volterra integral equations of convolution type, Ling [30], Miller [33], and Levin [27] have assumed that $a(t) > 0$ and $a'(t) < 0$. When we integrate (2.2.5) between 0 and t, then the resulting integral equation has the kernel $\int_0^t a(s)ds$ and hence the above assumptions cannot be satisfied for such a kernel. Therefore the results of [27], [30] and [33] are not applicable for (2.2.5). Our endeavour here has been to find conditions such that the auxiliary (or resolvent) kernel $b(t)$ is negative so that the solutions could be positive. More specially, our results assert that the resolvent could be negative without the kernel being negative.

2.5 Nonlinear Equations

Having known the conditions on the kernel $a(t)$ and the source function $f(t)$, for the solution $x(t)$ of the linear convolution equation (2.2.5), we shall now discuss similar

properties of solutions of the nonlinear equation

$$x'(t) = f(t)-\int_0^t a(t-s)x(s)ds+G(t,x(t)), x(0) = x_0 > 0 \quad (2.5.1)$$

where $0 \leq t < \infty$ under growth conditions on the nonlinear perturbation function $G(t,\varphi)$. Assume that

(H_1) : $G(t,\varphi) \geq 0$ for all $t \geq 0$ and $\varphi \geq 0$

(H_2) : there exists a positive continuous function $\lambda(t)$ such that

$$|G(t,\varphi)| \leq \lambda(t)\ (1+|\varphi|)$$

where $\lim_{t\to\infty} \lambda(t) = 0$, $0 \leq t < \infty$.

(H_3): the resolvent $\varphi(t)$ associated with the kernel $a(t)$ is positive for all $t \in [0,\infty)$ and $\varphi(t) \in L^1[0,\infty)$.

Theorem 2.5.1. Suppose the hypothesis (H_1), (H_2) and (H_3) are satisfied. If the solution $y(t)$ of (2.2.5) with $y(0) = x_0$ is positive and bounded on $[0,\infty)$, then every solution $x(t)$ of (2.5.1) with $x(0) = x_0$ satisfies the inequality

$$x_0 \leq x(t) < \infty$$

for all $0 \leq t < \infty$.

Proof. From Lemma (2.2.3), Remark (2.2.2) and (2.5.1), it follows that

$$x(t) = y(t) + \int_0^t \varphi(t-s)G(s,x(s))\ ds.$$

Define

$$H(s,x) = G(s,x_+) \text{ for } s \in [0,\infty),\ x \in \mathbb{R} \tag{2.5.2}$$

where $x_+ = x$ if $x \geq 0$

$= 0$ if $x < 0$.

Thus $H(s,x) \geq 0$ for $s \in [0,\infty)$; $x \in \mathbb{R}$. Let $\rho(t)$ be a solution of

$$\rho(t) = y(t) + \int_0^t \varphi(t-s)\, H(s,\rho(s))\, ds \tag{2.5.3}$$

Therefore, if $y(t)$ is positive on $[0,\infty)$, then (H_1), (H_3), (2.5.2) and (2.5.3) imply that $\rho(t)$ is positive on $[0,\infty)$ and hence the solution $x(t)$ of (2.5.1) is positive on $[0,\infty)$. Moreover, from the boundedness of $y(t)$ on $[0,\infty)$, the Corollary 2.1 of Nohel [37] yields that $x(t)$ is also bounded on $[0,\infty)$. This completes the proof.

2.6 Behaviour of Solutions over a Finite Interval

This section deals with the positivity of solutions defined over a finite interval, of Volterra integro-differential equations of both convolution and nonconvolution type. The generality of the conditions and the fruitfulness of the results are illustrated by means of examples.

Theorem 2.6.1

Suppose

(i) $0 < a(t) < \sigma$, σ being a constant

and

(ii) $f(t) > 0$ hold.

Then there exists an interval $[0,\alpha]$, where α is such that $(\sigma\alpha^2 + b\alpha) < 2$, b is a positive real number, on which every solution of (2.2.5) satisfies

$$0 < x(t) \leq x_0 + F(t)$$

where $F(t) = \int_0^t f(s)ds.$

Proof. From (2.2.11) we have

$$\Psi(t) = x_0 \bar{\varphi}(t) + \int_0^t f(s) \bar{\varphi}(t-s)ds.$$

For a given $\alpha > 0$ there exists a $b > 0$ such that $\sigma\alpha^2 + b\alpha < 2.$

Choose $\varphi(t) = -\frac{\sigma}{2}t^2 - \frac{b}{2} t+1$ for all $t \in [0,\alpha).$

Then $\bar{\varphi}(t) > 0$ $[0,\alpha]$ and $\varphi(0) = 1.$ Also $\varphi'(t)=\sigma(t)-b/2 < 0$ on $t \in [0,\alpha).$

Thus from (ii) we have $\Psi(t) > 0$ on $[0,\alpha]$.

Equation (2.2.10) gives

$$c(t) = \varphi'(0) + \int_0^t \bar{\varphi}(t-\tau)a(\tau)d\tau$$

This implies that $c(0) = \varphi'(0) = -b/2 < 0.$

Therefore from (i) and the fact $\varphi'(0) < 0$ for all $t \geq 0,$ it follows that

$$c'(t) = \varphi''(t) + \varphi(0)a(t) + \int_0^t \varphi'(t-\tau)a(\tau)d\tau$$

$$< -\sigma + \sigma + \int_0^t \varphi'(t-\tau)a(\tau)d\tau$$

$$< 0.$$

Thus we obtained $c(0) < 0$, $c'(t) < 0$ and hence $c(t) < 0$. Now from the observations made towards the end of section 2.1, $\Psi(t) > 0$, and $c(t) < 0$, every solution of (2.2.9) is positive. The application of Lemma 2.2.3 now yields that every solution $x(t)$ of (2.2.5) is positive.

Since $x(t)$ is positive on $[0,\alpha]$ and $a(t) > 0$ for $t \in [0,\alpha]$ we have from equation (2.2.5) that

$$x'(t) \leq f(t).$$

This in turn implies that $0 < x(t) \leq x_0 + F(t)$ for all $t \in [0,\alpha]$.

Theorem 2.6.2

Assume that

(h_1) : for $0 \leq s \leq t \leq T$

(i) $a(t,s) > 0$, $M_1 \leq a(s,s) \leq M_2$

(ii) $a_t(t,s) < 0$

(iii) $0 < a_s(t,s) < M_3$

(h_2) : for a given $\alpha (0 < \alpha \leq T)$ there exists positive constants β, γ, δ and λ such that

(i) $\beta > \gamma\alpha + \delta\alpha^2$

(ii) $\beta M_2 < \min \{\lambda\gamma, 2\delta\}$

(iii) $M_3 < \lambda M_1$

and

(h_3) :

(i) $\quad f(0) > 0$

(ii) $$f'(t) + \frac{\lambda(\gamma\alpha + \delta\alpha^2)}{(\delta\alpha^2+\gamma\alpha-\beta)} f(t) > \frac{x_o \; \beta\lambda(\gamma+ 2\delta\alpha)}{(\beta-\gamma\alpha-\delta\alpha^2)^2}$$

$\alpha, \beta, \gamma, \delta,$ and λ being the same constants as in (h_2).

Then on the interval $[0\ \alpha]$ every solution $x(t)$ of (2.1.2) satisfies the following inequalities

(I) $\quad 0 < x(t) < x_o + F(t)$

and

(II) $\quad 0 < x'(t) < f(t)$

where $x_o > 0$ and $F(t) = \int_0^t f(s)\,ds$.

Proof. For $0 \leq s \leq t < \alpha$ (fixed $\alpha \leq T$), define

$$\varphi(t,s) = p(t)\ q(s)$$

where $\quad p(t) = \dfrac{-\lambda}{-\beta+\gamma t^2+\delta t^2}\quad$ and $q(t) = (\beta - \gamma t - \delta t^2)$.

By (h_2) (i), $q(t) > 0$, $q'(t) < 0$ and $q''(t) < 0$ and $t \in [0,\alpha]$

(2.6.1)

Also $p(t) > 0$ and $p'(t) = \dfrac{\lambda(\gamma+2\delta t)}{(-\beta+\gamma\alpha+\delta\alpha^2)^2} > 0$ for $t \in [0,\alpha]$

(2.6.2)

Moreover $\varphi(t,t) = p(t)q(t) = \lambda > 0$.

From (2.2.2) and (h_2)(ii) we have

$$b(0,0) = a(0,0) + p(0)\, q'(0) < M_2 - \frac{\lambda\gamma}{\beta} < 0$$

Differentiating (2.2.2) with respect to 't' we get

$$b_t(t,s) = a_t(t,s) + p'(t)q'(s) - \int_s^t p'(t)q(u)a(u,s)\,du.$$

From (h_1)(i),(ii) (2.6.1) and (2.6.2) it follows that

$$b_t(t,s) < 0 \text{ on } t \in [0,\alpha] .$$

Differentiating (2.2.2) with respect to 's' we get

$$b_s(t,s) = a_s(t,s) + p(t)q''(s) + p(t)q(s)a(s,s) - \int_s^t p(t)q(u)a_s(u,s)\,du$$

$$- \lambda a(s,s) + \lambda \int_s^t a_s(u,s)\,du \qquad (2.6.3)$$

Since $p(t) > 0$, $q'(t) < 0$ and $a_s(t,s) > 0$ we have

$$\lambda \int_s^t a_s(u,s)\,du - \int_s^t p(t)q(u)a_s(u,s)\,du < 0 \text{ for all } 0 < s \leq t \leq \alpha \qquad (2.6.4)$$

Hypothesis (h_1), (h_2)(iii) now gives

$$a_s(t,s) - \lambda a(s,s) < M_3 - \lambda M_1 < 0 \qquad (2.6.5)$$

From (h_2) (ii) we have

$$0 > \beta M_2 - 2\delta > \beta a(s,s) - 2\delta > (\beta - \gamma s - \delta s^2)a(s,s) - 2\delta \qquad (2.6.6)$$

Further (2.6.6) together with $p(t) > 0$ gives

$$p(t)q''(s) + p(t)q(s)a(s,s) < 0 \qquad (2.6.7)$$

Now substituting (2.6.4), (2.6.5), (2.6.2) in (2.6.3) we get

$$b_s(t,s) < 0 \qquad \text{for } 0 \leq s \leq t \leq \alpha.$$

Since $b(0,0) < 0$, $b_t(t,s) < 0$ and $b_s(t,s) < 0$, it is clear that $b(t,s) < 0$.

From (2.2.3), (h_3)(i) and the fact $p'(t) > 0$, we have

$$h(0) = f(0) > 0$$

and

$$h(t) > f(t)+\lambda x_0-\beta x_0 p(t)+\lambda \int_0^t f(s)ds-p(\alpha)\int_0^t q(s)f(s)ds \tag{2.6.8}$$

Define the right side of the inequality (2.6.8) by $\chi(t)$, then $\chi(0) > 0$ and

$$\chi'(t) = f'(t)-\beta x_0 p'(t)+\lambda f(t) - \frac{\lambda}{\beta-\gamma\alpha-\delta\alpha^2}q(t)f(t) \tag{2.6.9}$$

Moreover from (h_3)(ii) we have

$$f'(t) +\lambda f(t) + \frac{\beta\lambda}{(\delta\alpha^2+\delta\alpha-\beta)} f(t) > \frac{x_0\beta\lambda(\gamma+2\delta\alpha)}{(\beta-\gamma\alpha-\delta\alpha^2)^2} > \frac{x_0\beta\lambda(\gamma+2\delta t)}{(\beta-\gamma t-\delta t^2)^2}.$$

This implies that

$$\begin{aligned} f'(t)+\lambda f(t) &> \beta x_0\, p'(t) - \frac{\beta\lambda}{(\delta\alpha^2+\gamma\alpha-\beta)} f(t) > \beta x_0\, p'(t) \\ &\quad - \frac{\lambda(\beta-\gamma t-\delta t^2)}{(\delta\alpha^2+\gamma\alpha-\beta)} f(t) \\ &\geq \beta x_0\, p'(t) + \frac{\lambda}{(\beta-\gamma\alpha-\delta\alpha^2)} q(t)f(t). \end{aligned} \tag{2.6.10}$$

Therefore (2.6.9) and (2.6.10) give $X'(t) > 0$ for $t \varepsilon [0,\alpha]$. This together with $X(0) > 0$ shows that $X(t) > 0$ for all $t \varepsilon [0,\alpha]$. Thus from (2.6.8) we have $h(t) > X(t) > 0$.

Finally the observations made towards the end of section of 2.1 ensures that all the solutions of (2.2.1) together with their derivatives are positive. Therefore in view of Lemma 2.2.1, the assertion of the theorem follows.

Remark 2.6.1

In theorem 2.6.2 the hypothesis (h_3) on f can be replaced by

$$(h_3^*) : (i)\ f(0) > x_o\ \lambda \left(\frac{\beta}{\beta-\gamma\alpha-\delta\alpha^2} - 1\right)$$

$$(ii)\ f'(t) > \lambda \left(\frac{\gamma\alpha + \delta\alpha^2}{\beta-\gamma\alpha-\delta\alpha^2}\right) f(t).$$

Example 2.6.1

Select $a(t,s) = e^{-t+s}$ for $0 \leq s \leq t \leq \alpha\ (= 1)$ and $x_o = 1$

Then $a(t,s) > 0$ and $a(s,s) = 1$. Therefore $M_1 = M_2 = 1$.

Moreover $a_t(t,s) = -e^{-t+s} < 0$ and $a_s(t,s) = e^{-t+s} > 0$ and $a_s(t,s) < 1\ (= M_3)$.

Let $\beta = 3/2$, $\gamma = 1/3$, $\delta = 1$ and $\lambda = 6$ so that (h_2) of theorem 2.6.2 is satisfied.

Choose $f(t) = e^{1000t}$, then $f(0) = 1$ and

$$f'(t) + \frac{\lambda(\gamma\alpha+\delta\alpha^2)}{(\delta\alpha^2+\gamma\alpha-\beta)} f(t) > 1000e^{1000t} - 48e^{1000t} > 952$$

$$> 756 = \frac{x_o\beta\lambda\ (\gamma+2\delta\alpha)}{(\beta-\gamma\alpha-\delta\alpha^2)^2} .$$

Thus all the conditions of theorem 2.6.2 are verified.

Example 2.6.2

If we take $f(t) = 50e^{50t}$ in example 2.6.1, then (h_3^*) holds. However, (h_3) does not hold. Therefore, the conditions (h_3) and (h_3^*) are some what independent.

CHAPTER - 3

EVENTUAL STABILITY PROPERTIES OF SOLUTIONS

3.1 Introduction

The purpose of this chapter is to study the eventual uniform stability and eventual uniform asymptotic stability of set $\{0\}$ for the integro-differential equation

$$x'(t) = f(t) + \int_0^t a(t,s)x(s)\, ds \qquad (3.1.1)$$

and the perturbed equation

$$z'(t) = f(t)+\int_0^t a(t,s)\{z(s)+g(s,z(s))\}ds. \quad (3.1.2)$$

Eventual uniform asymptotic stability is a natural generalisation of uniform asymptotic stability in which it is not assumed that the zero function is a solution of (3.1.1). Infact, this would happen if $f(t) \not\equiv 0$ in (3.1.1). The main tools used are the resolvent function and the variation of constants formula.

In the first part of this chapter, we shall investigate sufficient conditions for eventual uniform asymptotic stability of the set $\{0\}$ for (3.1.1) under certain hypotheses on the resolvent function $r(t,s)$ (corresponding to $a(t,s)$) and the weight function $f(t)$. Convolution equations are also discussed. In many practical problems if $\{0\}$ is eventually uniformly asymptotically stable for (3.1.1) then it is interesting to

know the class of perturbation functions g which preserve the same property for (3.1.2). In the later part of the chapter an attempt has been made in this direction under certain growth conditions on the nonlinear perturbation function g.

3.2 Definitions and Basic Results

The following definitions are useful in our subsequent discussion. Let $z(t,t_o,z(t_o))$ be any solution of (3.1.2) with $z(t_o,t_o z(t_o)) = z(t_o)$ existing for all $t \geq t_o$.

Definition 3.2.1.

The set {0} is eventually uniformly stable (EvUS) if for every $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon) \geq 0$ and $\delta = \delta(\varepsilon) > 0$ such that

$$|z(t,t_o,z(t_o))| < \varepsilon \text{ for all } |z(t_o)| < \delta \text{ and } t \geq t_o \geq \alpha.$$

Definition 3.2.2.

The set {0} is eventually uniformly attracting (EvUA) if there exists $\delta_o > 0$ and $\alpha_o \geq 0$ and if for every $\eta > 0$ there exists $T = T(\eta) \geq 0$ such that

$$|z(t,t_o,z(t_o))| < \eta \text{ for } |z(t_o)| < \delta_o, t_o \geq \alpha_o \text{ and } t \geq t_o+T$$

Definition 3.2.3.

The set {0} is eventually uniformly asymptotically stable (EvUAS) if it is both EvUS and EvUA.

Definition 3.2.4.

Let K_0 be the class of monotonic, non-negative functions $\lambda(.)$ defined on $[0,\infty)$ such that $\lambda(\rho) \to 0$ as $\rho \to 0^+$.

Definition 3.2.5.

Let K_∞ be the class of monotonic, non-negative functions $\nu(.)$ defined on $[0,\infty)$ such that $\nu(t) \to 0$ as $t \to \infty$.

The definitions 3.2.1 - 3.2.3 are equally applicable to the convolution equation

$$y'(t) = f(t) + \int_0^t b(t-s)y(s)ds \qquad (3.2.1)$$

and also to the nonconvolution equation (3.1.1).

We need the following basic result due to Grossman and Miller [16] expressing the solution x(t) of (3.1.1) in terms of the resolvent function r(t,s) satisfying the equation

$$\frac{\partial}{\partial s} r(t,s) = -\int_s^t r(t,u)a(u,s)du, \; r(t,t) = 1 \quad (3.2.2)$$

on the interval $0 \leq s \leq t$.

Let $LL^1(S)$ be the set of all measurable functions on a set S such that the seminorms are finite for all compact subsets of S.

Lemma 3.2.1 ([16])

Assume $a(t,s) \in LL^1(\mathbb{R} \times \mathbb{R}^+)$. Then the function r(t,s) satisfying the equation (3.2.2) exists on $0 \leq s \leq t$ and is continuous in (t,s). $\frac{\partial r(t,s)}{\partial s}$ exists a.e. on $0 \leq s \leq t$, is in

$LL^1(\mathbb{R}^+ \times \mathbb{R}^+)$. Moreover, given any $\bar{x}_o$ and a function $f(t) \in LL^1(\mathbb{R}^+)$ equation (3.1.1) is equivalent to

$$x(t) = r(t,0)\,\bar{x}_o + \int_0^t r(t,s)f(s)ds. \tag{3.2.3}$$

For the convolution equation (3.2.1) if $c(t)$ is the resolvent function satisfying the equation

$$c'(t) = \int_0^t b(t-s)c(s)ds,\; c(0) = 1 \tag{3.2.4}$$

then the solution $y(t)$ of (3.2.1) with $y(0) = \bar{y}_o$ is given by

$$y(t) = c(t)\,\bar{y}_o + \int_0^t c(t-s)\; f(s)ds. \tag{3.2.5}$$

We shall now state a result for volterra integro-differential equations which is analogous to lemma 3.1 of Strauss and Yorke [51].

Lemma 3.2.2.

Let $\{0\}$ be EvUAS for (3.1.1). Suppose there exists $\lambda \in K_o$ and $\nu \in K_\infty$ and for some $r_1 > 0$, each $m \in (0,r_1)$, and each $\tau > 0$, there exists $F_{\tau,m} \in K_\infty$ such that for each $\tau_1 \in (0,\tau]$, each $t_o \geq \alpha(r_1)$ and each solution $z(\cdot)$ of (3.1.2) satisfying

$$|z(t_o)| < \delta(r_1) \text{ and } m \leq |z(t)| \leq r_1 \text{ for } t_o \leq t \leq t_o + \tau,$$

there exists a solution $x(t)$ of (3.1.1) such that

$$|z(t)-x(t)| \leq \lambda(|z(t_o)|)\,\nu(t-t_o) + F_{\tau,m}(t_o)$$

for all $t_0 \leq t \leq t_0+\tau_1$. Then $\{0\}$ is EvUAS for (3.1.2).

To proof is virtually similar to that of Lemma 3.1 of Strauss and Yorke [51] and hence omitted.

3.3 Eventual Uniform Stability

In this section we shall investigate sufficient conditions for the eventual uniform stability of the Volterra integro-differential equation (3.1.1). A few propositions have been proved first which assert the integrability and boundedness properties of solutions of (3.1.1). These results have been used to obtain eventual uniform stability of the set $\{0\}$.

We shall now give an account of basic hypotheses on the resolvent function $r(t,s)$ and the weight function $f(t)$. The generality of the hypotheses and the fruitfulness of the results are indicated through various remarks and examples.

Assume that

(H_1) : $|r(t,s)| \leq M_1$ where M_1 is a positive constant,

(H_2) : $f(t) \in L^1[0,\infty)$ (i.e. $\int_0^\infty |f(s)|ds < M_2$, for some positive constant M_2).

Proposition 3.3.1.

If (H_1) and (H_2) are satisfied then all the solutions of (3.1.1) are bounded.

The proof is direct and follows from (3.2.3).

Remark 3.3.1.

The hypotheses (H_1) and (H_2) in Lemma 3.3.1 can be bartered for the following conditions :

(C_1) : $r(t,0)$ is bounded,

(C_2) : $\sup_{0\leq t<\infty} \int_{-1}^{t} \left(\int_{s}^{s+1} |r(t,\tau)|^q d\tau\right)^{1/q} ds < \infty,$

($r(t,s)$ is defined to be zero outside the region $0\leq s\leq t<\infty$.)

(C_3) : $\sup_{0\leq s<\infty} \left(\int_{s}^{s+1} |f(\tau)|^p \, d\tau\right) < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1.$

Indeed, it follows from (3.2.3) and Holder's inequality that

$$|x(t)| \leq |r(t,0)| \; |x_0| + \int_0^t |r(t,s)f(s)|ds$$

$$\leq M + \int_0^t \int_{\tau-1}^{\tau} |r(t,\tau)| \; |f(\tau)|ds \, d\tau ,$$

where M is some constant

$$\leq M + \int_{-1}^{t} \int_{s}^{s+1} |r(t,\tau)| \; |f(\tau)|d\tau \, ds$$

$$\leq M+\int_{-1}^{t} \left(\int_{s}^{s+1} |r(t,\tau)|^q d\tau\right)^{1/q}\left(\int_{s}^{s+1} |f(\tau)|^p \, d\tau\right)^{1/p} ds$$

$$\leq M + \sup_{0\leq s<\infty} \left(\int_{s}^{s+1} |f(\tau)|^p d\tau\right)^{1/p} \sup_{t\geq 0} \int_{-1}^{t} \left(\int_{s}^{s+1} |r(t,\tau)|^q d\tau\right)^{\frac{1}{q}} ds$$

$$< \infty.$$

Remark 3.3.2.

It is clear that (H_2) implies (C_3) but the converse is not true. Moreover (C_1) and (C_2) hold for $r(t,s) = e^{-(t-s)}$.

Proposition 3.3.2.

If (H_2) and the hypothesis

$(\tilde{H}_1)$: $c(t)$ is bounded (i.e. $|c(t)| \leq \bar{M}_1$ for some positive constant $\bar{M}_1$)

are satisfied, then all the solutions of (3.2.1) are bounded.

Proposition 3.3.3.

If (H_2) and the hypothesis

$$(H_3) : \sup_{0 \leq s < \infty} \int_s^\infty |r(t,s)|\,dt \leq M_3$$

are satisfied then all the solutions of (3.1.1) are in $L^1 [0,\infty)$.

Proof. From (3.2.3), (H_2) and (H_3) we have for any $T > 0$,

$$\int_0^T |x(t)|\,dt \leq \int_0^T |r(t,0)|\ |x_0|\,dt + \int_0^T \int_0^t |r(t,u)|\ |f(u)|\,du\,dt$$

$$\leq |x_0| \int_0^\infty |r(t,0)|\,dt + \int_0^T \left(\int_u^T |r(t,u)|\,dt\right) |f(u)|\,du$$

$$< \infty.$$

Remark 3.3.3.

The hypotheses (H_2) and (H_3) in proposition 3.3.3 can be replaced by

(C_4) : $\dfrac{f(t)}{X(t)}$ is $L^1 [0,\infty)$ and

$$(C_5) : \sup_{u \geq 0} \int_u^\infty |r(t,u)|\ X(t)\,dt < \infty$$

where $X(t)$ is any nonincreasing function greater than zero.

Remark 3.3.4.

In view of remark 3.3.1 the integrability condition on $f(t)$ may be relaxed to $\sup_{u \geq 0} \int_u^{u+1} |f(s)|^p \, ds < \infty$ in Proposition 3.3.3, if $r(t,s)$ satisfies (i) $r(t,0)$ is $L^1[0,\infty)$ and (ii) $\int_{-1}^{\infty} \left(\int_u^{u+1} \left(\int_s^{\infty} |r(t,s)|\,dt\right)^q ds\right)^{1/q} du < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ instead of (H_3).

Remark 3.3.5.

If $r(t,0)$ is integrable over $[0,\infty)$ and if

$$(C_6) : \int_0^{\infty} \int_0^{s} |r(s,\tau)f(\tau)|\,d\tau\, ds \text{ is finite}$$

then all the solutions of (3.1.1) are integrable over $[0,\infty)$. For example, if $r(t,s) = e^{-t-s}$ and $f(t) = e^t$ then (C_6) holds but (H_2) does not hold.

Proposition 3.3.4.

If the hypothesis (H_2) and

$$(\tilde{H}_3) : c(t) \in L^1[0,\infty)$$

are satisfied, then all the solutions of (3.2.1) are integrable over $[0,\infty)$.

Theorem 3.3.1

If the hypotheses (H_1), (H_2), (H_3) and

(H_4) : The function $\ell(s)$ defined by

$\ell(s) = \operatorname{ess\,Sup}_{0 \leq \tau \leq s} |a(s,\tau)|$ is such that

$\int_0^{\infty} \ell(s)\,ds \leq M_4$ where M_4 is some constant

are satisfied, then the set $\{0\}$ is eventually uniformly stable for (3.1.1).

Proof. Consider, for $0 \le t_0 < t$,

$$\int_{t_0}^{t} r(t,s)x'(s)ds = x(t)-r(t,t_0)x(t_0)-\int_{t_0}^{t} \frac{\partial}{\partial s}[r(t,s)]\ x(s)ds$$

Now (3.1.1) and the change of order of integration yield

$$\int_0^{t_0}\left(\int_{t_0}^{t} r(t,s)a(s,\tau)ds\right)x(\tau)d\tau + \int_{t_0}^{t}\left(\int_{\tau}^{t} r(t,s)a(s,\tau)ds\right)x(\tau)d\tau$$

$$+ \int_{t_0}^{t} r(t,s)f(s)ds + r(t,t_0)x(t_0) = x(t)-\int_{t_0}^{t} \frac{\partial}{\partial s}[r(t,s)]\ x(s)ds\ .$$

Invoking (3.2.2), we get

$$x(t) = r(t,t_0)x(t_0)+\int_{t_0}^{t} r(t,s)f(s)ds+\int_0^{t_0}\left(\int_{t_0}^{t} r(t,s)a(s,\tau)ds\right)x(\tau)d\tau \tag{3.3.1}$$

From (H_2), (H_3) and Proposition 3.3.3 we have

$\int_0^{\infty} |x(\tau)|d\tau \le M_5$ where M_5 is some positive constant.

For a given $\varepsilon > 0$, in view of (H_2) and (H_4) choose $T_1 = T_1(\frac{\varepsilon}{4M_1}) > 0$ and $T_2 = T_2(\frac{\varepsilon}{2M_1M_5}) > 0$ respectively, so large such that

$$\int_{T_1}^{t} |f(s)|ds < \frac{\varepsilon}{4M_1} \quad \text{for all } t \ge T_1 \tag{3.3.2}$$

and

$$\int_{T_2}^{t} \ell(s)ds < \frac{\varepsilon}{2M_1M_5} \text{ for all } t \ge T_2. \tag{3.3.3}$$

Define $\alpha(\varepsilon) = \max\{T_1(\frac{\varepsilon}{4M_1}),\ T_2(\frac{\varepsilon}{2M_1M_5})\}$ and $\delta(\varepsilon) = \frac{\varepsilon}{4M_1}$.

Then, for $t \geq t_o \geq \alpha(\varepsilon) \geq 0$ and $|x(t_o)| < \delta(\varepsilon)$, from (3.3.1), (3.3.2) and (3.3.3) we have

$$|x(t)| \leq |r(t,t_o)|\ |x(t_o)| + \int_{t_o}^{t} |r(t,s)|\ |f(s)|ds$$
$$+ \int_0^{t_o} (\int_{t_o}^{t} |r(t,s)|\ |a(s,\tau)|ds)\ |x(\tau)|d\tau$$
$$\leq M_1\ \delta(\varepsilon) + M_1 \frac{\varepsilon}{4M_1} + M_1M_5 \frac{\varepsilon}{2M_1M_5} = \varepsilon .$$

Thus the set {0} is eventually uniformly stable for (3.1.1).

Remark 3.3.6.

It can be easily seen that if (H_1), (H_2) and

(C_7) : for $s \leq t$, $u(t,s) = \int_0^s \int_s^t |r(t,\ell)a(\ell,\tau)|d\ell\ d\tau \to 0$ as $s \to \infty$

are satisfied, then the set {0} is eventually uniformly stable. In fact assertion still holds if we replace (C_7) by

$$v(t) = \int_0^t \int_t^\infty |a(s,\tau)|ds\ d\tau \to 0 \text{ as } t \to \infty.$$

3.4 Eventual Uniform Asymptotic Stability

In this section the eventual uniform asymptotic stability property of the set {0} for both convolution and non-convolution equations is discussed.

Theorem 3.4.1.

If the hypotheses (H_1), (H_2), (H_3), (H_4) and

(H_5) : $r(t,s) \to 0$ along the lines parallel to t-axis hold, then the set $\{0\}$ is eventually uniformly asymptotically stable for (3.1.1).

Proof. Since all the conditions of theorem 3.3.1 hold, the set $\{0\}$ is eventually uniformly stable for (3.1.1). Therefore, in view of definition 3.2.3 it is sufficient to show the eventual uniform attracting property of $\{0\}$. From the eventual uniform stability of the set $\{0\}$, we have

for a given $\eta > 0$ there exists $\delta(\eta) > 0$ and $\alpha(\eta) \geq 0$ such that

$$|x(t)| < \eta \quad \text{for all } t \geq t_1 \geq \alpha(\eta) \tag{3.4.1}$$

whenever $|x(t_1)| < \delta(\eta)$.

From (H_2), for a given $\frac{\delta(\eta)}{8M_1} > 0$ there exists $T_1\left(\frac{\delta(\eta)}{8M_1}\right) > 0$ such that

$$\int_{T_1\left(\frac{\delta(\eta)}{8M_1}\right)}^{t} |f(s)|ds < \frac{\delta(\eta)}{8M_1} \text{ for all } t \geq T_1\left(\frac{\delta(\eta)}{8M_1}\right). \tag{3.4.2}$$

Let $r(t,s_0) = \max\limits_{0 \leq s \leq T_1\left(\frac{\delta(\eta)}{8M_1}\right)} r(t,s)$ for all $t \geq T_1\left(\frac{\delta(\eta)}{8M_1}\right)$.

In view of (H_5), for a given $\frac{\delta(\eta)}{8M_2} > 0$ there exists $T_3\left(\frac{\delta(\eta)}{8M_2}\right) > 0$ such that

$$|r(t,s_0)| < \frac{\delta(\eta)}{8M_2} \text{ for all } t \geq T_3\left(\frac{\delta(\eta)}{8M_2}\right) \tag{3.4.3}$$

Using (3.4.2) and (3.4.3), for $t \geq N_1 = T_1\left(\frac{\delta(\eta)}{8M_1}\right) + T_3\left(\frac{\delta(\eta)}{8M_2}\right)$, we get

$$\int_{t_o}^{t} |r(t,s)|\ |f(s)|ds = \int_{t_o}^{T_1(\frac{\delta(\eta)}{8M_1})} |r(t,s)|\ |f(s)|ds$$

$$+ \int_{T_1(\frac{\delta(\eta)}{8M_1})}^{t} |r(t,s)|\ |f(s)|ds$$

$$\leq \frac{\delta(\eta)}{8M_2} M_2 + \frac{\delta(\eta)}{8M_1} M_1 = \frac{\delta(\eta)}{4} . \qquad (3.4.4)$$

Similarly, we obtain

$$\int_{t_o}^{t} |r(t,s)| \ell(s)ds < \frac{\delta(\eta)}{2M_5} \text{ for all } t \geq N_2 = T_3(\frac{\delta(\eta)}{4M_4M_5}) + T_2(\frac{\delta(\eta)}{4M_1M_5}) \qquad (3.4.5)$$

where $T_3(\xi)$ is obtained from (H_5) for any given $\xi > 0$. Further Proposition 3.3.1 asserts that there exists a constant $\rho > 0$ such that $|x(t)| \leq \rho$ for all $t \geq 0$. In view of eventual uniform stability of $\{0\}$, for $\rho > 0$ there exists $\delta(\rho) > 0$ and $\alpha(\rho) \geq 0$. Define $\delta_o = \delta(\rho)$ and $\alpha_o = \alpha(\rho)$. Let $0 < \eta < \rho$. Choose $T(\eta) = N_1+N_2+\alpha(\eta) + T_3(\frac{\delta(M)}{4\delta_o})$ and let $t_1 = t_o+T(\eta)$, where $t_o \geq \alpha_o \geq 0$.

From (3.3.1) we get

$$x(t_1) = r(t_1,t_o)x(t_o)+\int_{t_o}^{t_1} r(t_1,s)f(s)ds$$

$$+ \int_{0}^{t_o} (\int_{t_o}^{t_1} r(t_1,s)a(s,\tau)ds)x(\tau)\ d\tau \qquad (3.4.6)$$

Thus for $|x(t_o)| < \delta_o$, (3.4.6) together with Proposition 3.3.3, (H_4), (H_5), (3.4.4) and (3.4.5) gives

$$|x(t_1)| \leq \frac{\delta(\eta)}{4\delta_o} \quad \delta_o + \frac{\delta(\eta)}{4} + \frac{\delta(\eta)}{2M_5} M_5 = \delta(\eta).$$

Since $t_1 \geq \alpha(\eta)$, it follows from (3.4.1) that

$$|x(t)| < \eta \quad \text{for all } t \geq t_1 = t_o+T(\eta),\ t_o \geq \alpha_o.$$

Therefore the set {0} is eventually uniformly attracting for (3.1.1) and this proves the theorem.

Theorem 3.4.2.

If the hypotheses $(\tilde{H}_1)$, (H_2), $(\tilde{H}_3)$ and

$(\tilde{H}_4)$: $b(t) = b_1(t)+b_2(t)$ where $b_1(t)$ is bounded and $b_2(t)$ is integrable over $[0,\infty)$

are satisfied, then the set {0} is eventually uniformly asymptotically stable for (3.2.1).

Proof. As in the proof of theorem 3.3.1, from (3.2.1), (3.2.4) and for all $t \geq t_o$, we have

$$y(t) = c(t-t_o)y(t_o)+\int_{t_o}^{t} c(t-s)f(s)ds+\int_{0}^{t_o}\left(\int_{t_o}^{t} c(t-s)b(s-\tau)ds\right)y(\tau)d\tau \tag{3.4.7}$$

Consider the integral

$$\left|\int_{0}^{t_o}\left(\int_{t_o}^{t} c(t-s)b(s-\tau)ds\right)y(\tau)d\tau\right|$$

$$\leq \int_{t_o}^{t}\left(\int_{0}^{t_o}|b(s-\tau)|\ |y(\tau)|d\tau\right)|c(t-s)|ds$$

$$\leq \int_{t_o}^{t}\left(\int_{0}^{s}|b(s-\tau)|\ |y(\tau)|d\tau\right)|c(t-s)|ds$$

$$\leq \int_{t_o}^{t}\left(\int_{0}^{t-s}|b_1(t-s-\tau)|\ |y(\tau)|d\tau\right)|c(s)|\ ds$$

$$+\int_{0}^{\infty}\Psi(s)|\ c(t-s)|ds \tag{3.4.8}$$

where $\Psi(s) = \int_{0}^{s} |b_2(s-\tau)|\ |y(\tau)|d\tau$.

From $(\tilde{H}_3)$, Proposition 3.3.4 and $(\tilde{H}_4)$, we have for a given $\varepsilon > 0$ there exists a $\hat{T}_3(\varepsilon) > 0$ such that

$$\int_{t_0}^{t} \left(\int_{0}^{t-s} |b_1(t-s-\tau)|\ |y(\tau)|d\tau\right) |c(s)|ds < \frac{\varepsilon}{4} \text{ for } t \geq t_0 \geq \hat{T}_3 \tag{3.4.9}$$

In view of hypotheses and equation (3.2.4) it is clear that $c'(t)$ is bounded and hence $c(t)$ is uniformly continuous. Also by $(\tilde{H}_4)$ and Proposition 3.3.4, $\Psi(s) \in L^1[0,\infty)$ and is positive. Since $c(t)$ is uniformly continuous and $\Psi(t)$ is $L^1[0,\infty)$, $\int_{0}^{\infty} \Psi(s)\ |c(t-s)|ds$ is uniformly continuous and moreover it is integrable because $\Psi(s)$ is $L^1[0,\infty)$ and $(\tilde{H}_3)$. Now the application of a lemma of Barbalat [cf. 13, p. 89] yields that $\int_{0}^{\infty} \Psi(s)\ |c(t-s)|\ ds \to 0$ as $t \to \infty$.

Therefore for a given $\varepsilon > 0$ there exists a $\hat{T}_2(\varepsilon) > 0$ such that

$$\int_{0}^{\infty} \Psi(s)\ |c(t-s)|ds < \frac{\varepsilon}{4} \text{ for } t \geq t_0 \geq \hat{T}_2. \tag{3.4.10}$$

By (H_2) for a given $\varepsilon > 0$, there exists $T = T_1(\frac{\varepsilon}{4\bar{M}_1}) > 0$ such that

$$\int_{T_1}^{t} |f(s)|\ ds < \frac{\varepsilon}{4\bar{M}_1} \text{ for all } t \geq T_1. \tag{3.4.11}$$

Define $\alpha(\varepsilon) = \max\{\hat{T}_3, \hat{T}_2, T_1(\frac{\varepsilon}{4\bar{M}_1})\}$ and $\delta(\varepsilon) = \frac{\varepsilon}{4\bar{M}_1}$.

Then for $t \geq t_0 \geq \alpha(\varepsilon) > 0$ and $|y(t_0)| < \delta(\varepsilon)$, equations (3.4.7) - (3.4.11) give

$$|y(t)| \leq \bar{M}_1 \ \delta(\varepsilon) + \bar{M}_1 \frac{\varepsilon}{4\bar{M}_1} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Hence the set $\{0\}$ is eventually uniformly stable for (3.2.1). Since $c(t)$ is uniformly continuous and is integrable, $c(t) \to$ as $t \to \infty$.

Now proceeding as in the proof of Theorem 3.4.1, it can be easily seen that the set $\{0\}$ is eventually uniformly asymptotically stable for (3.2.1).

3.5 Perturbed Equations

The eventual uniform asymptotic stability of the set $\{0\}$ for (3.1.2) is discussed in this section under the class of perterbations that satisfy the following hypothesis

$$(H_6) : |g(t,z)| \leq \lambda(t) \ (1+|z|) \text{ for } t \geq 0, \ |z| < \infty$$
$$\text{where } \lambda(t) \in L^1 [0,\infty).$$

Theorem 3.5.1.

Let the hypothesis of Theorem 3.4.1 together with (H_6) be satisfied. Then the set $\{0\}$ is eventually uniformly asymptotically stable for (3.1.2).

Proof. Hypotheses (H_1) and (H_4) imply that

$$\left|\frac{\partial}{\partial s} r(t,s)\right| < M_6 \qquad (3.5.1)$$

where M_6 is some positive constant.

From the variation of constants formula, the solution $z(t)$ of (3.1.2) can be expressed as

$$z(t) = x(t)+r(t,t_o)(z(t_o)-x(t_o))$$
$$+ \int_0^{t_o} \left(\int_{t_o}^{t} r(\tau,u)a(u,\tau)du\right) g(\tau,z(\tau))d\tau$$
$$- \int_{t_o}^{t} \frac{\partial}{\partial\tau} r(t,\tau)\ g(\tau,\ z(\tau))d\tau$$
$$+ \int_0^{t_o} \left(\int_{t_o}^{t} r(t,u)a(u,\tau)du\right)(z(\tau)-x(\tau))\ d\tau, \qquad (3.5.2)$$

where t_o is some arbitrary point and $x(t)$ is a solution of the linear equation (3.1.1). For $t_o = 0$ and for $z(0) = x(0)$, the equation (3.5.2) takes the form

$$z(t) = x(t) - \int_0^{t} \frac{\partial}{\partial\tau} r(t,\tau)\ g(\tau,\ z(\tau))d\tau\ ,$$

then the hypothesis (H_6) together with (3.5.1) gives

$$|z(t)| < M_7+M_6 \int_0^{t} \lambda(\tau)\ |z(\tau)|\ d\tau$$

where M_7 is a positive constant.

Since $\lambda(t) \in L^1[0,\infty)$, the application of Gronwall-Bellman inequality yields that every solution $z(t)$ of (3.1.2) is bounded. If $x(t)$ is a solution of (3.1.1) such that $x(t_o) = z(t_o)$ then (3.5.2) together with (3.5.1), (H_1), (H_4) and (H_6) gives

$$|z(t)-x(t)| \leq p(t,t_o) + q(t,t_o) \int_0^{t} |z(\tau)-x(\tau)|\ d\tau \qquad (3.5.3)$$

where $p(t,t_o) = M_8\left(\int_{t_o}^{t} \ell(u)du\right) + M_9 \int_{t_o}^{t} \lambda(u)\ du$ and

$$q(t,t_o) = M_1(\int_{t_o}^{t} \ell(u)du),\ M_8, M_9 \text{ being positive constants.}$$

It can be clearly seen that $p(t,t_o)$ and $q(t,t_o)$ are bounded and increasing in t. Further, $p(t,t_o), q(t,t_o) \to 0$ as $t_o \to \infty$. Applying Gronwall-Bellman type inequality [cf. 53,p.16] to (3.5.3) we obtain

$$|z(t)-x(t)| \le p(t,t_o)+q(t,t_o)(\int_0^t p(s,t_o)ds)e^{\int_0^t q(s,t_o)ds}$$

$$\le p(\tau,t_o)+q(\tau,t_o)N_1\tau\ e^{N_2\tau} \quad \text{for } t \in [0\ \tau),$$

where N_1, N_2 are bounds on $p(t,t_o)$ and $q(t,t_o)$ respectively. Let $F_{\tau,m}(t_o) = p(\tau,t_o)+q(\tau,t_o)N_1\tau\ e^{N_2\tau}$. Then, it is clear that $F_{\tau,m}(t_o) \to 0$ as $t_o \to \infty$.

Thus the hypotheses of Lemma 3.2.2 hold with $\lambda(\cdot) \equiv 0$ and hence the set $\{0\}$ is EvUAS for (3.1.2).

CHAPTER - 4

LIAPUNOV-RAZUMIKHIN TECHNIQUE

4.1 Introduction

Liapunov second method or the direct method involving an energy like function has come to stay as a powerful tool in the qualitative study of ordinary differential equations. Over the years, this method has also been extended to functional differential equations (FDE) by various authors. While Liapunov functions are employed in the study of ordinary differential equations, more generally Liapunov **functionals are** used in studying FDE (**cf.** [19],[54]). As Volterra integro-differential equations (VIDE) can also be treated as FDE, such a study can be carried over to these equations. Further Liapunov functionals have been constructed exclusively for VIDE by Burton ([1] , [2]) in order to study the stability and uniform stability properties of VIDE. As distinguished from this line, Razumikhin [41] obtained stability properties of FDE using Liapunov functions rather than functionals. This technique of Razumikhin involves an estimation of the derivative of the Liapunov function along the solutions of the system of equations on a certain set in which the solutions satisfy a specified inequality. In recent years a good number of papers (cf. [15],[21],[24],[47],[48]) have appeared employing this technique and various results have been accumulated. In

particular, Seifert [47],[48] and Grimmer and Seifert [15] have studied the properties of solutions of VIDE such as stability, boundedness, asymptotic stability, uniform stability and so on, by studying the corresponding properties of FDE. In this chapter Liapunov-Razumikhin technique is further explored to yield L^2-stability of the zero solution of VIDE. This has been achieved as an application of a more general result on FDE which has been derived in section 3.

Section 2 deals with the preliminary results that are necessary for proving the main results. For completeness some of the results of Seifert are included in this section.

4.2 Preliminaries and basic results

In this section the employed notations are explained and then it is indicated how a VIDE can be thought of as an ordinary differential equation involving an interval of delay which becomes unbounded as $t \to +\infty$. In addition two theorems of Seifert [47] are given which ensure the stability of the point $x = 0$ and boundedness of the solutions, for (4.2.1).

For an element x of $\mathbb{R}^n$, by $|x|$ we mean the usual euclidian norm. We denote by $x_t(\cdot)$ a function continuous on the interval $0 \leq s \leq t$ to $\mathbb{R}^n$ and by S_t we mean the set $\{x_t(\cdot)\}$ of all such functions. If $x(s)$ is a function defined and continuous on $0 \leq s < \infty$ to $\mathbb{R}^n$, then for each fixed t, $0 \leq t < \infty$, this function defines a member $x_t(\cdot)$ of S_t given by $x(s)$, $0 \leq s \leq t$. We call this function $x_t(\cdot)$ a segment of $x(s)$.

For fixed $t \geq 0$, let $F(t,x_t(\cdot))$ be a function on S_t to $\mathbb{R}^n$. For each function $x(s)$ continuous on $0 \leq s < \infty$ to $\mathbb{R}^n$, we assume that $F(t,x_t(\cdot))$ is continuous in t, where $x_t(\cdot)$ is a segment of $x(s)$.

By a solution of the equation

$$x'(t) = F(t,x_t(\cdot)) \tag{4.2.1}$$

we mean a continuously differentiable function $x(s)$ on $0 \leq s < \infty$ such that (4.2.1) is satisfied on $0 \leq t < \infty$ for $x_t(\cdot)$ a segment of $x(s)$. Let $x(t,x_o)$ be any solution of (4.2.1) existing for all $t \geq 0$ such that $x(0,x_o) = x_o$.

Definition 4.2.1.

The point $x = 0$ for (4.2.1) is stable if given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $|x_o| < \delta(\varepsilon)$, then every solution $x(t,x_o)$ of (4.2.1) is defined for $t \geq 0$ and satisfies $|x(t,x_o)| < \varepsilon$ for $t \geq 0$.

Definition 4.2.2.

The point $x = 0$ is L^p-stable, $0 < p < \infty$, for (4.2.1) if it is stable and if there exists a $\delta_o > 0$ such that $\int_0^\infty |x(t,x_o)|^p \, dt < \infty$ whenever $|x_o| < \delta_o$.

We shall now consider the VIDE of the form

$$x'(t) = H(x(t)) + \int_0^t G(t,s,x(s))ds, \quad x(0) = x_o \tag{4.2.2}$$

for $t \geq 0$. Here $H(x)$ is continuous on $\mathbb{R}^n$, and $G(t,s,x)$ is

continuous on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n$.

It can benoted that if $x(t)$ is a solution of (4.2.2), then it is also a solution of (4.2.1) where

$$F(t,\xi(.)) = H(\xi(t)) + \int_0^t G(t,s,\xi(s))ds$$

and $x_t(.)$ is a segment of $x(s)$, $0 \leq s < \infty$.

We shall now state the following results due to Seifert [47] which are useful in our subsequent discussion.

Theorem 4.2.1.

Let there exists functions $u(s)$, $v(s)$, and $f(s)$ continuous for $s \geq 0$ and such that $u(0) = v(0) = 0$, $u(s)$ is increasing, $f(s) > s$ for $s > 0$, and suppose $V(t,x)$ is a real-valued function continuous in (t,x) for $t \geq 0$ and x in D, an open subset of $\mathbb{R}$ containing the zero vector. Let V satisfy :

(i) $u(|x|) \leq V(t,x) \leq v(|x|)$ for $t \geq 0$, x in D;

(ii) $V'(t,x(t)) \leq 0$ for any solution $x(t)$ of (4.2.1) for which $x(s)$ is in D and $f(V(t,x(t))) > V(s,x(s))$ for $0 \leq s \leq t$. Then the point $x = 0$ is stable for (4.2.1).

Remark 4.2.1.

In Theorem 4.2.1 and onwards, by $V'(t,x(t))$ we mean

$$\lim_{h\to 0+} \operatorname{Sup} \frac{[V(t+h,x(t+h)) - V(t,x(t))]}{h}.$$

Theorem 4.2.2.

Let there exist a function V satisfying the hypotheses of Theorem 4.2.1 except that now $D = \mathbb{R}^n$ and $u(r) \to \infty$ as $r \to \infty$. Then the solutions of (4.2.1) are bounded.

4.3 L^p-Stability of Functional Differential Systems

In this section, we shall first obtain a general result yielding L^p-stability of the point zero for the FDE (4.2.1). Then this result has been applied to derive L^2-stability properties of the zero solution of an integro-differential system in Section 4.4.

Theorem 4.3.1.

Suppose $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is a continuous function and satisfy the following conditions :

(i) $u(|x|) \leq V(t,x) \leq v(|x|)$ for all (t,x)

where $u,v : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions such that $u(0) = 0 = v(0)$ and $u(s)$ is increasing to $+\infty$ as s increases to $+\infty$.

(ii) For any solution $x(t)$ of (4.2.1)

$$V'(t,x(t)) \leq - C|x(t)|^p , \quad p > 0, \ C > 0,$$

whenever $f(V(t,x(t))) > V(s,x(s))$ for s in any neighbourhood $(t-\eta,t)$, $\eta > 0$, and where $f(s)$ is any continuous function for $s \geq 0$ such that $f(s) > s$ for $s > 0$.

Then the point $x = 0$ is L^p-stable for (4.2.1).

Proof. Define

$$r(t) = V(t,x(t)) + C \int_0^t |x(t,x_0)|^p \, dt$$

Clearly $r(0) = V(0,x_0)$ and $r(t)$ is positive.

We claim that $r(t) \leq r(0)$ for $t \geq 0$. Suppose not. Then there exists a $\tilde{t}_1 > 0$ such that $r(\tilde{t}_1) > r(0)$. Since $r(t)$ is a continuous function of t, there exists a $t_1 \geq 0$ and a $t_2 > t_1$ such that $r(t) \leq r(0)$ on $[0,t_1]$, $r(t_1) = r(0)$ and $r(t) > r(0)$ on (t_1,t_2). Therefore for some $t^* \varepsilon (t_1,t_2)$ there exists an $\eta > 0$ such that

$$r'(s) > 0 \text{ for } s \ \varepsilon \ (t^*-\eta, t] \ . \qquad (4.3.1)$$

Further for $t^* - \eta < s < t^*$,

$$r(s) < r(t^*). \qquad (4.3.2)$$

From hypotheses and Theorem 4.2.2 it follows that

$$\sup_{0 \leq t < \infty} |x(t,x_0)| < M_1(x_0)$$

where $M_1(x_0)$ is a positive constant and $x(0,x_0) = x_0$. From continuous dependence of solutions on the initial values we get $M_1(x_0) \leq M$ for $|x_0| < 1$, where M is a constant independent of x_0.

Hence the monotonocity of v yields

$$v(|x(t,x_0)|) \leq v(M), \text{ for } t \geq 0, \ |x_0| \leq 1.$$

Let ε be given. We may assume that ε so small that $u(\varepsilon) < v(M)$. Then there is a positive number $a = a(\varepsilon)$ such that

$$f(s) > s+a \quad \text{for } s \in [u(\varepsilon), v(M)] . \tag{4.3.3}$$

From the hypotheses and Theorem 4.2.1 it is clear that the point $x = 0$ is stable for (4.2.1). (4.3.4)

Therefore, for a given $L = (\frac{a}{2\eta C})^{1/p}$ there exists a $\tilde{\delta}_0 > 0$ such that

$$|x_0| < \tilde{\delta}_0 \text{ implies } |x(t,x_0)| < (\frac{a}{2\eta C})^{1/p}, \; t \geq 0.$$

In particular for $s \in (t^*-\eta, t^*)$ and for $|x_0| < \delta_0 = \min\{\tilde{\delta}_0, 1\}$, we have

$$|x(s,x_0)|^p < \frac{a}{2\eta C} .$$

This implies that

$$C \int_s^{t^*} |x(\tau,x_0)|^p \, d\tau < \frac{Ca}{2\eta C} (t^*-s)$$

$$< \frac{a}{2} < a \tag{4.3.5}$$

From (4.3.2) and the definition of $r(t)$, we get

$$V(t^*,x(t^*)) + C \int_s^{t^*} |x(\tau,x_0)|^p \, d\tau > V(s,x(s)) \tag{4.3.6}$$

For a given $\varepsilon > 0$ there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ such that $v(\varepsilon) < u(\delta_1)$. Since $x = 0$ of 4.3.1 is stable, it is clear that $u(\delta_1) \leq V(t,x(t)) \leq v(M)$. Hence from (4.3.3), we have

$$f(V(t^*,x(t^*))) > V(t^*,x(t^*)) + a$$

$$> V(t^*,x(t^*)) + C \int_s^{t^*} |x(\tau,x_0)|^p \, d\tau$$

(from 4.3.5)

$$> V(s,x(s)) \qquad \text{(from 4.3.6)}$$

Thus condition (ii) of hypotheses gives

$$V'(t^*,x(t^*)) < - C|x(t^*)|^p$$

which in turn implies $r'(t^*) < 0$ contradicting (4.3.1). Therefore, there exists no $\tilde{t}_1 > 0$ such that $r(\tilde{t}_1) > r(0)$. Hence $r(t) \leq r(0)$ for all $t \geq 0$. Thus we have

$$V(t,x(t)) + C \int_0^t |x(\tau,x_0)|^p \, d\tau < V(0,x_0) \text{ and therefore}$$

$$\int_0^t |x(\tau,x_0)|^p \, d\tau < \frac{1}{C} V(0,x_0) \text{ for } |x_0| < \delta_0.$$

This together with (4.3.4) proves the theorem.

4.4 Integro-Differential Systems

As an application of Theorem 4.3.1, we consider an integro-differential system

$$x'(t) = Ax(t)+h(t,x(t))+\int_0^t g(t,s,x(s))ds, \; x(0) = x_0 \tag{4.4.1}$$

where A is a real nxn constant matrix, h is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and satisfying

$$|h(t,x)| \leq \mu |x|, \tag{4.4.2}$$

and $g(t,s,x)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n$ for $t \geq s \geq 0$, and satisfies

$$|g(t,s,x)| \leq L(t,x)\ |x| \tag{4.4.3}$$

with $\int_0^t L(t,s)ds \to 0$ as $t \to \infty$.

Theorem 4.4.1.

Assume that

(i) A is a stable matrix,

(ii) (4.4.2) and (4.4.3) hold.

Then for sufficiently small μ , $x = 0$ of (4.4.1) is L^2-stable.

Proof. Since A is stable, it is clear (cf. [44]) that there exists a positive definite symmetric matrix B such that

$$A^T B + BA = -I$$

where A^T is the transpose of A and I is the identity matrix. Define $V = x^T Bx$. Let α and β be respectively the smallest and largest eigenvalues of B. Then we have

$$\alpha^2\ |x|^2 \leq x^T B\ x \leq \beta^2\ |x|^2 \tag{4.4.4}$$

for $x \in \mathbb{R}^n$. Hence the condition (i) of Theorem 4.3.1 is verified. To show that the condition (ii) of Theorem 4.3.1 holds for μ sufficiently small, let

$$\mu < \frac{\alpha}{2|B|\beta} \tag{4.4.5}$$

where $|B| = \sum_{i,j=1}^{n} |b_{ij}|$, $B = (b_{ij})$.

Choose $f(s) = q^2 s$ where $q > 1$ is such that

$$\mu < \frac{\lambda}{2q|B|\beta} \tag{4.4.6}$$

Then for any $\eta > 0$ and any solution $x(t)$ of (4.4.1) such that

$$x^T(s)\ Bx(s) < q^2 x^T(t)\ Bx(t) \text{ for } s \in [t-\eta, t], \tag{4.4.7}$$

we have

$$q^2\beta^2|x(t)|^2 > \alpha^2|x(s)|^2 \text{ for all } t \geq \eta. \tag{4.4.8}$$

For a solution $x(t)$ of (4.4.1) with $V(t,x(t)) = x^T(t)Bx(t)$, it follows for $t \geq 0$ that

$$\dot{V}(t,x(t)) \leq -|x(t)|^2 + 2|B|\ |x(t)|\ |h(t,x(t))|$$

$$+ 2|B|\ |x(t)| \int_0^t |g(t,s,x(s))|ds. \tag{4.4.9}$$

Fix $\mu_1 > 0$ such that

$$\left(\frac{2|B|q\beta}{\alpha}\right)(\mu + \mu_1) < 1. \tag{4.4.10}$$

In view of (4.4.3) there exists a $\eta > 0$ such that for $t \geq \eta$,

$$\int_0^t L(t,x)ds \leq \mu_1.$$

Since

$$\int_{t-\eta}^t |g(t,s,x(s))|\ ds \leq \int_{t-\eta}^t L(t,x)\ |x(s)|ds$$

$$\leq \sup_{\theta\varepsilon[t-\eta,t]} x(\theta) \int_0^t L(t,s)ds \tag{4.4.12}$$

From (4.4.11) and (4.4.12), for a solution $x(t)$ of (4.4.1) satisfying (4.4.8), we have

$$2|B|\;|x(t)|\int_{t-\eta}^{t}|g(t,s,x(s))|ds \leq \frac{2q\beta\mu_1|B|}{\alpha}\;|x(t)|^2 \quad (4.4.13)$$

Further (4.4.3) gives

$$\int_{0}^{t-\eta}|g(t,s,x(s))|\;ds \leq M\int_{0}^{t}L(t,s)ds \to 0 \text{ as } t \to \infty \quad (4.4.14)$$

Thus (4.4.9), (4.4.10), (4.4.13) and (4.4.14) lead to assumption (ii) of Theorem 4.3.1. Hence the application of Theorem 4.3.1 yields the desired result.

CHAPTER - 5

ANALYSIS OF LINEAR SYSTEMS

5.1 Introduction

The study of linear systems is vital in order to penetrate into the complexity of non-linear systems. Besides, the linear systems themselves are very often fitted as convenient mathematical models for various physical phenomena. Thus it is important to accumulate more information regarding the behavioural pattern of linear systems. This chapter has been exclusively devoted for the analysis of linear systems.

It is well known that, a linear autonomous ordinary differential system is asymptotically stable if all the eigenvalues of the corresponding coefficient matrix have negative real parts (cf. [44, Chapter 3]). For non-autonomous systems, with an addition of Lipschitz condition on the coefficient matrix, similar results have been expounded in [4],[5]. Thus while studying Volterra integro-differential systems (VIDS), be it through Lyapunov second method (cf. [1], [2], [47],[48]) or from perturbation analysis (cf. [42], [44]) it has invariably been assumed that the coefficient matrix is stable. But this amounts to requiring the stability of the corresponding ordinary differential equation. Notable exceptions that have dispensed with the stability condition on the coefficient matrix have been the works of Levin [25],

Grossman and Miller [17], Grimmer and Seifert [15], and Burton [3]. In [25] this has been done by defining a suitable energy function while in [17] the integrability of the resolvent function of VIDS has been characterised by a transformation condition similar to that given in [39] for Volterra integral equations. In [15], the same has been achieved by studying the resolvent of a transformed equation. Quite recently in [3] the conditions involving the anti-derivatives of the kernel are assumed. Motivated by the interesting nature of this problem and the above cited works, an attempt has been made in Section 2 to study the asymptotic behaviour of solutions of VIDS when the coefficient matrix is not necessarily stable. Our approach has been through an equivalent VIDS involving an arbitrary function satisfying certain assumptions and thus paving the way for the new coefficient matrix (corresponding to equivalent VIDS) to be stable.

The results obtained in Section 2 have been used to study a linear ordinary differential system and the conditions for integrability of its solutions when they are bounded and vice versa are discussed in Section 3.

A line of thought in recent years has been that the dichotomies rather than Liapunov characteristic exponents form a proper base for the study of asymptotic behaviour of solutions of non-autonomous differential systems. Massera and Schäffer [32], Coppel [5,6,7], Saker and Sell [46] among

othershave discussed dichotomy properties of ordinary differential systems and several results have been reported. These properties have been extended for Volterra integro-differential systems in Section 4.

5.2 Stability properties of solutions

In this section it is proposed to study the stability properties, more specifically the exponential stability, of the solution x(t) of

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds + F(t), \tag{5.2.1}$$

wherein A(t) and K(t,s) are nxn matrices defined and continuous over $0 \leq t < \infty$ and $0 \leq s \leq t < \infty$ respectively and x(t) and F(t) are n-vectors with F(t) continuous over $0 \leq t < \infty$, when the matrix A is not necessarily stable. Our main approach is by way of an equivalence theorem (Lemma 5.2.1) that has the potential to supply us with a stable matrix B corresponding to A. The efficiency of this approach has been demonstrated through a suitable example.

The following lemmas are useful in our subsequent discussion.

Lemma 5.2.1

If $\phi(t,s)$ is any continuously differentiable function over $0 \leq s \leq t < \infty$, then equation (5.?.1) is equivalent to

$$x'(t) = B(t)x(t) + \int_0^t L(t,s)\ x(s)ds + H(t) \tag{5.2.2}$$

where $$B(t) = A(t) - \Phi(t,t),$$

$$L(t,s) = K(t,s) + \Phi_s(t,s) + \Phi(t,s)A(s) + \int_s^t \Phi(t,u)K(u,s)du, \tag{5.2.3}$$

and

$$H(t) = F(t) + \Phi(t,0)x_0 + \int_0^t \Phi(t,s)F(s)ds.$$

Proof. Let $x(t)$ be any solution of (5.2.1) existing on the interval $0 \leq t < \infty$. Consider the identity

$$\int_0^t \Phi_s(t,s)x(s)ds = \Phi(t,t)x(t)-\Phi(t,0)x_0 - \int_0^t \Phi(t,s)x'(s)ds.$$

Substituting for $x'(t)$ from (5.2.1) and using Fubinis theorem, we get

$$\int_0^t \Phi_s(t,s)x(s)ds = \Phi(t,t)x(t) - \Phi(t,0)x_0-\int_0^t \Phi(t,s)F(s)ds - \int_0^t \Phi(t,s)A(s)x(s)ds - \int_0^t (\int_\tau^t \Phi(t,s)K(s,\tau)ds)x(\tau)d\tau \tag{5.2.4}$$

Therefore, it follows from (5.2.1) - (5.2.4) that

$$\int_0^t L(t,s)x(s)ds = \int_0^t K(t,s)x(s)ds + \int_0^t \Phi_s(t,s)x(s)ds + \int_0^t \Phi(t,s)A(s)x(s)ds + \int_0^t (\int_s^t \Phi(t,u)K(u,s)du)x(s)ds$$

$$= x'(t) - A(t)x(t)-F(t)+\Phi(t,t)x(t)-\Phi(t,0)x_0 - \int_0^t \Phi(t,s)F(s)ds$$

$$= x'(t) - B(t)x(t) + H(t) \text{ and } x(0) = x_0.$$

Thus every solution of (5.2.1) is also a solution of (5.2.2). Conversely, if $y(t)$ is any solution of (5.2.2) then by taking $\Phi(t,s) \equiv 0$ we have from (5.2.2) and (5.2.3), $L(t,s) = K(t,s)$ and $H(t) = F(t)$ and hence $y(t)$ satisfies (5.2.1).

Remark 5.2.1.

It can be readyly observed that when $\Phi(t,s)$ is the differential resolvent corresponding to the kernel $K(t,s)$, then the equation (5.2.2) together with (5.2.3) gives the usual variation of constants formula (See Grossman and Miller [16]).

Lemma 5.2.2.

If $B(t)$ is any nxn matrix function which commutes with its integral and satisfies the condition

$$\left| e^{\int_s^t B(\tau)d\tau} \right| \leq M\, e^{-\alpha(t-s)}, \quad 0 \leq s \leq t, \qquad (5.2.5)$$

then the solution $x(t)$ of (5.2.2) with $x(0) = x_0$ satisfies

$$|x(t)| \leq M|x_0|\, e^{-\alpha t} + M \int_0^t e^{-\alpha(t-\tau)}\, |H(\tau)|\, d\tau$$

$$+ M \int_0^t \left(\int_s^t e^{-\alpha(t-\tau)}\, |L(\tau,s)|\, d\tau \right) |x(s)|\, ds. \qquad (5.2.6)$$

Proof. Multiplying both sides of equation (5.2.2) by $\exp\left(-\int_0^t B(\tau)d\tau\right)$ and rearranging the terms, we obtain

$$\left(e^{-\int_0^t B(\tau)d\tau} x(t)\right)' = e^{-\int_0^t B(\tau)d\tau} \left[H(t) + \int_0^t L(t,s)x(s)ds\right] .$$

(v) $|\Phi_s(t,s)| \le N\, e^{-\delta(t-s)}$, N, δ are positive constants

(vi) for this Φ, the hypotheses of Lemma 5.2.2 hold,

and

(vii) $\gamma > \beta > \alpha$, $\delta > \alpha$ and $\alpha_o < \alpha$ where

$$\alpha_o \overset{\text{def}}{=\!=} \left[\frac{MJ}{(\beta-\alpha)} + \frac{MN}{(\delta-\alpha)} + \frac{MLA_o}{(\gamma-\alpha)} + \frac{MLJ}{(\beta-\alpha)(\gamma-\beta)}\right] .$$

Then every solution $x(t)$ of (5.2.1) tends to zero exponentially as $t \to \infty$.

Proof.

In view of lemma 5.2.1 and for a function Φ satisfying the conditions (iv), (v) and (vi), it is enough if we show that every solution of the equation (5.2.2) tends to zero exponentially as $t \to \infty$. Since $F(t) \equiv 0$, equations (5.2.3), (5.2.6) and condition (iv) imply that

$$e^{\alpha t}\,|x(t)| \le M|x_o| + ML|x_o| \int_o^t e^{-(\gamma-\alpha)\tau}\, d\tau$$

$$+ M \int_o^t \Big(\int_s^t e^{\alpha\tau} |K(\tau,s)+\Phi_s(\tau,s) + \Phi(\tau,s)A(s)$$

$$+ \int_s^\tau \Phi(\tau,u)\, K(u,s)\,du|\,d\tau\Big)\,|x(s)|\,ds$$

$$\le M|x_o| + \frac{ML\,|x_o|}{(\gamma-\alpha)}\left[1 - e^{-(\gamma-\alpha)t}\right]$$

$$+ M \int_o^t \Big(\int_s^t e^{\alpha\tau} |K(\tau,s)|\, d\tau\Big)\,|x(s)|\, ds$$

$$+ M \int_o^t \Big(\int_s^t e^{\alpha\tau}\, |\Phi_s(\tau,s)|\,d\tau\Big)\,|x(s)|\,ds$$

$$+ M \int_0^t \left(\int_s^t e^{\alpha\tau} \, |\Phi(\tau,s)| \, |A(s)| \, d\tau \right) |x(s)| ds$$

$$+ M \int_0^t \left[\int_s^t e^{\alpha\tau} \left(\int_s^\tau |\Phi(\tau,u)| \, |K(u,s)| du \right) d\tau \right] |x(s)| ds.$$

Further, from (ii) and (vii) we have

$$e^{\alpha t}|x(t)| \leq M|x_0| + \frac{ML|x_0|}{(\gamma-\alpha)} + M \int_0^t \left(\int_s^t e^{\alpha(\tau-s)} |K(\tau,s)| d\tau \right) e^{\alpha s} |x(s)| ds$$

$$+ M \int_0^t \left(\int_s^t e^{\alpha(\tau-s)} \, |\Phi_s(\tau,s)| d\tau \right) e^{\alpha s} \, |x(s)| ds$$

$$+ MA_0 \int_0^t \left(\int_s^t e^{\alpha(\tau-s)} |\Phi(\tau,s)| d\tau \right) e^{\alpha s} |x(s)| ds$$

$$+ M \int_0^t \left[\int_s^t e^{\alpha(\tau-s)} \left(\int_s^\tau |\Phi(\tau,u)| \, |K(u,s)| du \right) d\tau \right]$$

$$e^{\alpha s} \, |x(s)| ds. \qquad (5.2.7)$$

Now we shall estimate each integral on the right side of (5.2.7).

Using condition (iii), we get

$$\int_s^t e^{\alpha(\tau-s)} \, |K(\tau,s)| \, d\tau$$

$$\leq J \, e^{(\beta-\alpha)s} \int_s^t e^{-(\beta-\alpha)\tau} \, d\tau$$

$$= \frac{J}{(\beta-\alpha)} \left[1 - e^{-(\beta-\alpha)(t-s)}\right] . \qquad (5.2.8)$$

Condition (ii) gives

$$\int_s^t e^{\alpha(\tau-s)} \, |\Phi_s(\tau,s)| \, d\tau$$

$$\leq N \, e^{(\delta-\alpha)s} \int_s^t e^{-(\delta-\alpha)\tau} \, d\tau = \frac{N}{(\delta-\alpha)} \left[1-e^{-(\delta-\alpha)(t-s)}\right]$$

$$(5.2.9)$$

Similarly, estimating the other integrals by using (iii) and (iv), we obtain

$$\int_s^t e^{\alpha(\tau-s)} \ |\Phi(\tau,s)|\,d\tau \le \frac{L}{(\gamma-\alpha)} \ [1 - e^{-(\gamma-\alpha)(t-s)}] \tag{5.2.10}$$

and $\int_s^t e^{\alpha(\tau-s)}\left(\int_s^\tau |\Phi(\tau,u)|\ |K(u,s)|\,du\right)d\tau$

$$\le \frac{LJ}{(\beta-\alpha)(\gamma-\alpha)} - \frac{LJ}{(\gamma-\beta)(\beta-\alpha)}\, e^{-(\beta-\alpha)(t-s)}$$

$$+ \frac{LJ}{(\gamma-\beta)(\gamma-\alpha)}\, e^{-(\gamma-\alpha)(t-s)} \tag{5.2.11}$$

The inequalities (5.2.8) - (5.2.11) and (5.2.7) lead to

$$e^{\alpha t}|x(t)| \le M|x_0| + \frac{ML|x_0|}{(\gamma-\alpha)} + \int_0^t \Big[\frac{MJ}{(\beta-\alpha)} + \frac{MN}{(\delta-\alpha)} + \frac{MLA_0}{(\gamma-\alpha)}$$

$$+ \frac{MLJ}{(\beta-\alpha)(\gamma-\alpha)} + \frac{MLJ}{(\gamma-\beta)(\gamma-\alpha)}\, e^{-(\gamma-\alpha)(t-s)}\Big]\, e^{\alpha s}|x(s)|\,ds$$

$$\le M|x_0|\ \Big[1 + \frac{L}{(\gamma-\alpha)}\Big] + \int_0^t \Big[\frac{MJ}{(\beta-\alpha)} + \frac{MN}{(\delta-\alpha)} + \frac{MLA_0}{(\gamma-\alpha)}$$

$$+ \frac{MLJ}{(\beta-\alpha)(\gamma-\beta)}\Big]\, e^{\alpha s}\ |x(s)|\,ds$$

$$\le M|x_0|\ \Big[1 + \frac{L}{(\gamma-\alpha)}\Big] + \int_0^t \alpha_0\ e^{\alpha s}|x(s)|\,ds.$$

Applying Gronwall-Bellman inequality we obtain

$$e^{\alpha t}\ |x(t)| \le M|x_0|\ \Big(1 + \frac{L}{(\gamma-\alpha)}\Big)\ e^{\alpha_0 t}\ .$$

This implies that

$$|x(t)| \leq M|x_o| \left(1 + \frac{L}{(\gamma-\alpha)}\right) e^{(\alpha_o-\alpha)t} .$$

Thus in view of condition (vii), the result follows.

Remark 5.2.3.

It is possible to select a matrix function $\Phi(t,s)$ satisfying conditions (iv), (v) and (vi) of theorem 5.2.1. For example if $\Phi(t,s) = L\, e^{-\gamma(t-s)} I$, then $N = \frac{L}{\gamma}$ and $\delta = \gamma$. $\Phi(t,t)$ being a constant matrix in this case, the estimate (5.2.5) is guaranteed if $A(t)$ is a constant matrix and B is negative definite.

Remark 5.2.4

Basically it is the condition "$\alpha_o < \alpha$" which controls the asymptotic nature of the solution $x(t)$. A look at the composition of α_o reveals that while so choosing γ and δ much away from α we can nullify the effect of the last three terms in α_o, the first term $\left[\frac{MJ}{\beta-\alpha}\right]$ is the essential term with which we have to reckon with. Therefore if β is so large as to exceed $(\alpha^2+MJ)/\alpha$, then only α_o would be less than α. Thus we see that the attenuation required on the kernel $K(t,s)$ is linked with the constant α in (5.2.5). This conclusion implicitly assumes the estimate (5.2.5). Such an estimate is always possible when the transformed matrix B is constant and negative definite.

Remark 5.2.5.

If $F(t)$ is not zero in theorem 5.2.1, still the solutions of (5.2.1) tend to zero provided $F(t)$ is integrable. This is an immediate consequence of variation of constants formula (cf. [16]) and theorem 5.2.1.

Remark 5.2.6.

In [3], a condition of the type (5.2.5) has been used for the matrix $Q \stackrel{\text{def}}{=\!=} [A(t) - G(t,t)]$, where $G(t,s)$ is the antiderivative of the kernel $K(t,s)$ (i.e. $\frac{\partial G(t,s)}{\partial t} = K(t,s)$) As such the matrix B in our study allows more flexibility due to the arbitrary character of the function $\phi(t,s)$. Further, our approach is entirely different and the analysis in [3] can be applied to the equation (5.2.2) in order to obtain sharper estimates. Thus our theorem 5.2.1 is in addition to the theorem 2 of [3] rather than a substitute for it.

Example 5.2.1

In (5.2.1) (scalar case), let $K(t,s) = e^{-10(t-s)}$, $A(t) = 10e^{-55t} - 5$ and $F(t) \equiv 0$. Choose $\phi(t,s) = 10e^{-55t}$, then $\alpha = 5$ and $\alpha_0 < 3.25$. Thus all the conditions of Theorem 5.2.1 are satisfied.

5.3 An ordinary differential equation

In this section, an ordinary differential equation is considered as a special case of Volterra integro-differential equation (5.2.1) where the kernel $K(t,s) \equiv 0$. Lemma 5.2.1 has

been made use of to study L^1-like properties of solutions of this equation when they are given bounded and vice versa.

Consider the equation

$$x'(t) = A(t)x(t), \; x(0) = x_0 \qquad (5.3.1)$$

Invoking Lemma 5.2.1 with $K(t,s) \equiv 0$ and $F(t) \equiv 0$, the system (5.3.1) is equivalent to

$$x'(t) = [A(t)-\Phi(t,t)] \; x(t) + \int_0^t [\Phi_s(t,s)+\Phi(t,s)A(s)] \; x(s)ds + \Phi(t,0)x_0 \; .$$

This in turn reduces to

$$\Phi(t,t)x(t) = \Phi(t,0)x_0 + \int_0^t [\Phi_s(t,s)+\Phi(t,s)A(s)] \; x(s)ds. \qquad (5.3.2)$$

Further, if we select the function $\Phi(t,s)$ such that $\Phi(t,t)$ is non-singular for each $t \geq 0$, then (5.3.2) takes the form

$$x(t) = G(t) + \int_0^t U(t,s)x(s)ds, \qquad (5.3.3)$$

where $\quad G(t) = \Phi^{-1}(t,t) \;\; \Phi(t,0)x_0$

and

$$U(t,s) = \Phi^{-1}(t,t) \;\; [\Phi_s(t,s) + \Phi(t,s)A(s)] \; .$$

Clearly (5.3.3) is in the form of Volterra integral equation.

Theorem 5.3.1.

If the matrix function $\Phi(t,s)$ is such that

(i) $U(t,t)$ is non-singular for each $t \geq 0$

$$\text{(ii)} \quad \sup_{0 \le t < \infty} \int_0^t |U(t,s) - U(t,t)| \, ds < \infty$$

and $$\text{(iii)} \quad \sup_{0 \le t < \infty} |G(t)| < \infty,$$

then every bounded solution $x(t)$ of (5.3.1) satisfies

$$\sup_{0 \le t < \infty} U(t,t) \left| \int_0^t x(s) ds \right| < \infty \qquad (5.3.4)$$

Proof. From (5.3.3), we have

$$x(t) - U(t,t) \int_0^t x(s) ds = G(t) + \int_0^t [U(t,s) - U(t,t)] \, x(s) ds$$

and hence

$$\left| x(t) - U(t,t) \int_0^t x(s) ds \right| \le |G(t)| + \int_0^t |U(t,s) - U(t,t)| \, |x(s)| \, ds.$$

This together with the hypotheses gives the result.

Theorem 5.3.2.

If $x(t)$ is any L^1-solution of (5.3.1) and if $\Phi(t,s)$ is a matrix function such that

(i) $U(t,t)$ is bounded

(ii) $\sup_{0 \le s \le t < \infty} |U(t,s) - U(t,t)| < \infty$

and

(iii) $\sup_{0 \le t < \infty} |G(t)| < \infty$, then

$x(t)$ is bounded.

The proof is similar to that of theorem 5.3.1 and hence omitted.

Remark 5.3.1

For a choice of $\Phi(t,s) = e^{-t+s}I$, the conditions of Theorem 5.3.1 reduces to

(i) $(A(t)+I)$ is nonsingular,

(ii) $\sup_{0\leq t<\infty} |\int_0^t e^{-t+s} A(s)ds| < \infty$ and

(iii) $\sup_{0\leq t<\infty} t\left[A(t)+I\right] < \infty$ respectively.

For the same choice of $\Phi(t,s)$, the assumptions (i) and (ii) of theorem 5.3.2 respectively takes the form

(i) $(A(t)+I)$ is bounded and

(ii) $\sup_{0\leq s\leq t<\infty} e^{-t+s} A(s) < \infty$.

Example 5.3.1

Consider the scalar differential equation

$$x' = -\frac{t}{1+t}\, x. \tag{5.3.5}$$

Take $\Phi(t,s) = e^{-t+s}$, then all the conditions of Theorems 5.3.1 and 5.3.2 are satisfied. The solution $x(t)$ of (5.3.5) with $x(0) = x_0$ is given by $x(t) = x_0(1+t)e^{-t}$ which is integrable and bounded.

5.4 Dichotomies of solutions

While studying the stability and asymptotic stability properties of autonomous linear differential equations, we find that these properties are basically characterised by the sign of the real parts of the eigen values of the coefficient matrix. However an automatic generalization of this approach does not carry over to non-autonomous equations for studying the corresponding properties of uniform stability and uniform asymptotic stability. An example indicating this failure has been given by Hoppensteadt [20]. Therefore for such equations the study of dichotomies assumes importance.

In this section after giving necessary definitions and a lemma, the dichotomy properties of systems of Volterra integro-differential equations are studied.

The solution x(t) (cf. Grossman and Miller [19]) of the system

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds, \quad x(0) = x_0 \tag{5.4.1}$$

is given by $x(t) = R(t,0)x_0$ (5.4.2)

where the nxn matrix function $R(t,s)$, $0 \leq s \leq t < \infty$, is the resolvent corresponding to the functions $A(t)$ and $K(t,s)$.

Lemma 5.4.1

$R(t,0)$ is invertible for each $t \geq 0$.

Proof Suppose not, then $\exists$ a $t_1 > 0$ such that $R(t_1,0)$ is singular. Therefore the system of equations $R(t_1,0)x = 0$ has a non-zero vector, say $\bar{x}$, as a solution. If $x(0) = x_o$ is any initial point of the solution then by (5.4.2) we have

$$x(t_1) = R(t_1,0)\ x_o \tag{5.4.3}$$

Also if $\tilde{x} = x_o + \bar{x}$ then $\tilde{x} \neq x_o$ and

$$R(t_1,0)\tilde{x} = R(t_1,0)x_o + R(t_1,0)\bar{x} = x(t_1)+0 = x(t_1) \tag{5.4.4}$$

Thus (5.4.3) and (5.4.4) indicate violation of uniqueness of solutions of (5.2.1) and hence a contradiction. Thus the lemma is proved.

Definition 5.4.1

System (5.4.1) is said to possess an exponential dichotomy if there exists a projection matrix P and positive constants M, N, α, β such that the following conditions are satisfied.

$$\begin{aligned} &|R(t,0)\ PR^{-1}(s,0)| \le M\ e^{-\alpha(t-s)} \text{ for } t \ge s \\ \text{and } &|R(t,0)(I-P)R^{-1}(s,0)| \le N\ e^{-\beta(s-t)} \text{ for } s \ge t \end{aligned} \tag{5.4.5}$$

Definition 5.4.2

System (5.4.1) is said to possess bounded growth on an interval $J = \mathbb{R}^+$ if, for some fixed $r > 0$, there exists a constant $C \ge 1$ such that every solution $x(t)$ of (5.4.1) satisfies

$$|x(t)| \leq C|x(s)| \text{ for } 0 \leq s \leq t \leq s+r$$

Lemma 5.4.2.

System (5.4.1) has bounded growth if and only if there exists constants L, γ such that its resolvent matrix satisfies

$$|R(t,0)R^{-1}(s,0)| \leq L\, e^{\gamma(t-s)} \text{ for } t \geq s \qquad (5.4.6)$$

Proof : Suppose (5.4.1) has bounded growth. Then for $s \leq t \leq s+r$

$$|R(t,0)R^{-1}(s,0)| = \sup_{|x(s)|=1} |R(t,0)R^{-1}(s,0)x(s)|$$

$$= |x(t)| \leq L$$

where L = C.

Similarly for $s + r \leq t \leq s+2r$

$$|R(t,0)\; R^{-1}(s,0)| \leq L^2$$

and in general for $t \geq s$

$$|R(t,0)\; R^{-1}(s,0)| \leq e^{L} < L\, e^{L}$$

If we take $\gamma = \frac{L}{r}$, then for $t \geq s+r$,

$$L \leq \gamma(t-s)$$

and therefore $|R(t,0)\; R^{-1}(s,0)| \leq L\, e^{\gamma(t-s)}$.

Conversely, when (5.4.6) is true, we can have bounded growth of the system by taking $C = \max\ \{1, L\, e^{\gamma r}\}$

Remark 5.4.1

Lemma 5.4.2 shows that the condition for bounded growth is independent of the choice of r.

Remark 5.4.2.

The conditions (5.4.5) for exponential dichotomy can be expressed in the following equivalent form.

$$|R(t,0)\ P\xi| \le M\ e^{-\alpha(t-s)}|R(s,0)P\xi| \text{ for } t \ge s,$$
$$|R(t,0)(I-P)\xi| \le N\ e^{-\beta(t-s)}|R(s,0)(I-P)\xi| \text{ for } s \ge t, \tag{5.4.7}$$

and $|R(t,0)\ PR^{-1}(t,0)| \le M$ for all $t \ge 0$.

Lemma 5.4.3.

If (5.4.1) has exponential dichotomy on a subinterval $[t_o,\infty)$ then it also has exponential dichotomy on the half line $\mathbb{R}^+$, with the same projection P and the same exponents α and β.

Proof : Choosing $W = \sup_{0\le\tau\le t_o} |R(\tau,0)|$

we have $|R(t,0)\ R^{-1}(s,0)| \le W$ for $0 \le s,\ t \le t_o$.

If $0 \le s \le t_o \le t$, then

$$\begin{aligned}|R(t,0)PR^{-1}(s,0)| &= |R(t,0)PR^{-1}(t_o,0)R(t_o,0)R^{-1}(s,0)|\\ &= W|R(t,0)\ PR^{-1}(t_o,0)|\\ &\le WMe^{-\alpha(t-t_o)} \le WMe^{\alpha t_o}e^{-\alpha(t-s)}\end{aligned}$$

If $0 \le s \le t \le t_o$ then

$$|R(t,0)PR^{-1}(s,0)| = |R(t,0)R^{-1}(t_o,0)R(t_o,0)PR^{-1}(t_o,0)$$

$$R(t_o,0)\ R^{-1}(s,0)|$$

$$\leq W^2\ |R(t_o,0)\ PR^{-1}(t_o,0)|$$

$$\leq W^2M \leq W^2M\ e^{\alpha t_o}\ e^{-\alpha(t-s)}\ .$$

Hence $|R(t,0)PR^{-1}(s,0)| \leq \tilde{M}\ e^{-\alpha(t-s)}$ for $0 \leq s \leq t < \infty$, where

$$\tilde{M} = \max\ (WMe^{\alpha t_o},\ W^2M\ e^{\alpha t_o}).$$

Similarly, $|R(t,0)(I-P)\ R^{-1}(s,0)| \leq \tilde{L}\ e^{-\beta(t-s)}$ for $0 \leq t \leq s < \infty$.

Now we shall prove a result which indicates that bounded growth leads to exponential dichotomy.

Theorem 5.4.1.

Suppose that every solution $x(t)$ of (5.4.1) satisfies the following conditions

$$|x(t)| \leq C|x(s)| \quad \text{for } 0 \leq s \leq t \leq s+T$$

$$(5.4.8)$$

and $\quad |x(t)| \leq \theta \sup_{|u-t|\leq T} |x(u)|$ for $t \geq T$

where T, C, θ are constants such that $T > 0$, $C > 1$ and $0 < \theta < 1$

Then equation (5.4.1) has exponential dichotomy on $\mathbb{R}^+$.

Proof : Supposing that $x(t)$ is a non-trivial bounded solution then define for $s > 0$,

$$\lambda(s) = \sup_{u \geq s} |x(u)|.$$

Then for $t \geq s+T$

$$|x(t)| \leq \theta \sup_{|u-t|\leq T} |x(u)| \leq \theta \lambda(s)$$

and therefore $\lambda(s) = \sup_{s\leq u\leq s+T} |x(u)|$.

Further, it follows that

$$|x(t)| \leq C\, |x(s)| \quad \text{for } 0 \leq s \leq t < \infty.$$

If $s+nT \leq t < s+(n+1)T$, then

$$|x(t)| \leq \theta^n \sup_{|u-t|\leq nT} |x(u)| \leq \theta^n C |x(s)|$$

$$\leq \theta^{-1} C\, \theta^{\frac{t-s}{T}} |x(s)|.$$

Thus $|x(t)| \leq M\, e^{-\alpha(t-s)} |x(s)|$ for $0 \leq s \leq t < \infty$

where $M = \frac{C}{\theta} > 1$ and $\alpha = -\frac{\ln \theta}{T} > 0$.

Now suppose that $x(t)$ is an unbounded solution with $|x(0)| = 1$. Then there exist a sequence $\{t_n\}$, $t_n > 0$ such that

$$|x(t_n)| = \frac{C}{\theta^n}, \quad |x(t)| < \frac{C}{\theta^n} \text{ for } 0 \leq t < t_n .$$

Then $T < t_1 < t_2 < \ldots$ and $t_n \to \infty$ as $n \to \infty$ and moreover $t_{n+1} \leq t_n + T$, since $|x(t_n)| \leq \theta \sup_{0\leq u\leq t_n+T} |x(u)|$ and $|x(u)| < \theta^{-1} |x(t_n)|$ for $0 \leq u < t_{n+1}$.

Suppose $t \leq s$ and $t_m \leq t < t_{m+1}$, $t_n \leq s < t_{n+1}$ $(1 \leq m \leq n)$, then

$$|x(t)| \leq \theta^{-m-1} C = \theta^{n-m} |x(t_{n+1})|$$
$$\leq C\,\theta^{-1}\,\theta^{n-m+1} |x(s)|$$
$$\leq C\,\theta^{-1}\,\theta^{\frac{(s-t)}{T}} |x(s)|.$$

Thus $|x(t)| \leq M\, e^{-\alpha(s-t)} |x(s)|$ for $t_1 \leq t \leq s < \infty$, where $\alpha = -\frac{\ln\,\theta}{T}$ and $M = \frac{C}{\theta}$.

Now let V_1 be the subspace of the vector space $\mathbb{R}^n$ consisting of the initial values of all bounded solutions of (5.4.1), and let V_2 be subspace complimentary to V_1. For any vector $\xi \varepsilon V_2$ such that $|\xi| = 1$, let $x(t) = x(t,\xi)$ be the solution which takes the value ξ at $t = 0$. Then $x(t,\xi)$ is unbounded and therefore there exists a least value $t_1 = t_1(\xi)$ such that $|x(t_1,\xi)| = \frac{C}{\theta}$. We claim that the values $t_1(\xi)$ are bounded. Suppose not then there exists a sequence of unit vectors $\xi_\nu \varepsilon V_2$ such that $t_1(\xi_\nu) \to \infty$ as $\nu \to \infty$. By the compactness of the unit sphere in V_2 we may suppose that $\xi_\nu \to \xi$, where $|\xi| = 1$. Then

$$x(t,\xi_\nu) \to x(t,\xi) \text{ for every } t \geq 0.$$

Since $|x(t,\xi_\nu)| < \frac{C}{\theta}$ for $0 \leq t < t_1(\xi_\nu)$ it follows that

$$|x(t,\xi)| \leq \frac{C}{\theta} \text{ for } 0 \leq t < \infty$$

which is a contradiction to the fact that $\xi \varepsilon V_2$. Thus there exists $T_1 > 0$ such that $t_1(\xi) \leq T_1$ for all ξ, and therefore every solution $x(t)$ with $x(0) \varepsilon V_2$ satisfies

$$|x(t)| \leq M\, e^{-\alpha(s-t)}\, |x(s)| \text{ for } T_1 \leq t \leq s < \infty.$$

Consequently the first two conditions of (5.4.7) are satisfied on the interval $[T_1, \infty)$ and third condition is obvious. Therefore (5.4.1) has an exponential dichotomy on $[T_1, \infty)$. Thus in view of lemma 5.4.3 the exponential dichotomy of (5.4.1) is guaranteed on $\mathbb{R}^+$.

Remark 5.4.3.

For ordinary differential equations a result similar to theorem 5.4.1. has been proved by Massera and Schaffer [32].

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